A Generalized Algorithm for Duration and Convexity of Option Embedded Bonds

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Abstract

This article derives a generalized algorithm for duration and convexity of option embedded bonds that provides a convenient way of estimating the dollar value of 1 basis point change in yield known as DV01, an important metric in the bond market. As delta approaches 1, duration of callable bonds approaches zero once the bond is called. However, when the delta is zero, the short call is worthless and duration of callable will be equal to that of a straight bond. On the other hand, the convexity of a callable bond follows the same behavior when the delta is 1 as shown in Dunetz and Mahoney (1988) as well as in Mehran and Homaifar's (1993) derivations. However, in the case when the delta is zero, the convexity of a callable bond approaches zero as well, which is in stark contrast to the non-zero convexity derived in Dunetz and Mahoney's paper. Our generalized algorithm shows that duration and convexity nearly symmetrically underestimate (overestimate) the actual price change by 11/10 basis points for +/- 100 basis points change in yield. Furthermore, our algorithm reduces to that of MH for convertible bonds assuming the convertible bond is not callable.

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Recent innovations in duration and convexity to the valuation of bonds have been on option-free bonds. However, most bonds issued in the market have option embedded features such as callable, convertible, extendable, puttable, and other options to name a few. Dunetz and Mahoney (1988) extended duration and convexity analysis to callable bonds. Mehran and Homaifar (1993) hereafter MH derive duration and convexity for convertible bond as well as formulating an optimum call policy for the issuers of convertibles. This study extends MH algorithm to a generalized framework in which duration and convexity can be estimated for a callable convertible as well as any other option embedded bonds. The price dynamics of option-embedded bonds differs from that of a pure bond, as the embedded options reflect the price of the underlying stock as well as the changes in the market interest rate. Convertibles possess positive convexity as noted by Yawitz (1988). In contrast, callable bonds have a negative convexity as documented in Fabbozzi (2012), as this class of bonds is callable in the falling interest rate environment that is advantageous to the issuer and disadvantageous to the investors. Therefore, cash flows and maturities of the option-embedded bonds are uncertain and thus, using simple duration and convexity of pure bonds is an inadequate proxy for the duration and convexity of callable convertibles.

Expanding bond price with respect to change in yield using Taylor expansion, we get the following expression.

$$dP = \partial P/\partial K \text{ x } dK + 1/2!\ \partial P^2/\partial K^2 \text{ x } dK^2 + \ldots\ldots \qquad 1$$

The first derivative of the bond with respect to changes in yield to maturity normalized by the price of a bond is the modified duration of the bond.

$$Dm = \partial P / \partial K \times 1/P \qquad 2$$

Modified duration is the price elasticity of the bond with respect to change in yield. Modified duration measures how sensitive is the bond to change in yield for macroeconomic or firm-specific reasons, as it is a very useful parameter in bond portfolio management. The first term in the Taylor series can be positive or negative for any basis points (bps) changes in the yield, say plus or minus 25 bps. The second term in the Taylor series is equal to the second derivative of the bond price with respect to change in yield normalized by the price of a bond and is known as the convexity of the bond.

$$C = \partial P^2 / \partial K^2 \times 1/P \qquad 3$$

The above term is always positive for plus/minus incremental changes in yield, due to the square term in the Taylor series dK^2 of equation 1.

Convertible bonds can be callable, puttable, and be converted to something other than common stock. That means, there are two non-offsetting calls; one granted to investors and one granted to the issuer. Both investors and issuers have different motivations for exercising the options. For example, investors are motivated by profit as they are likely to convert the bonds into common stocks when it economically pays off to do so. This means investors forgo the coupon interest in the convertible bond in order to have an ownership interest in the issuing company's common stocks that may or may not pay a dividend. This scenario while exposing investors to higher risk, but also is likely to provide higher reward as well as future growth opportunities in the event the stock continues to go up, giving investors capital gains that are subject to lower tax as opposed to ordinary income in the form of coupon interest. On the other hand, issuers by granting themselves short call may have different sets of motivations in forcing conversion which may include.

1. Issuers may force the conversion whether or not such action is consistent with the objective of maximizing market value. Issuer my force the conversion in order to window dress the balance sheet as debt is converted to equity altering debt to equity ratio. The window dressing may be intended to reduce the amount of debt in the company's balance sheet in order to receive a higher rating on its debt. We call this scenario a window dressing hypothesis to be tested empirically.

2. Issuer my force the conversion, before the expiration of stock options granted to senior managers. The issuer faces a moral hazard problem as well as conflict of interest in forcing the conversion that may make no economic sense. This scenario is related to agency problem arising between the creditors and stockholders; issuing convertibles is likely to mitigate agency cost.

3. We believe window dressing is by far the most reasonable explanation for the issuance of convertibles as 2900 out of 3000 convertibles in table 1 are NR (not

rated). That means over 96.67 percent of all convertibles are issued by up and coming small companies in dire need of a relatively cheaper form of funding. The remaining 100 issues are rated from high of AA- to low of CC+. Only 38 out of 3000 issues of convertibles are rated investment grade that is BBB and higher.

4. Table 2 details various forms of convertible bonds. Nearly half of all convertibles (1469 out of 3000) issues are simply convertibles as a portfolio of long bond and long call. There are 216 issues of convertible callable, 344 convertible perpetual, 561 convertible puttable, 257 issues of convertible/put/call, and 136 issues of convertible/put/perp to name a few. The rest are in a double and single digit.

The providers of capital; shareholders and creditors, respectively face different payoff functions. Shareholders face a convex payoff function, while creditors face a concave pay off function because of agency related problems. Therefore, it is possible that shareholders to transfer wealth from creditors by increasing risk. Issuing convertible bonds may mitigate this agency related problem through risk shifting once the debt is converted into equity. Dorion et al (2014) analyze when risk shifting is high, increasing the incentive to issue convertibles. Stock price reaction is negative after the announcement of issuing convertibles, as shares are expected to dilute in the future.

Generally, financially distressed firms with a poor credit rating issue convertibles. The issuer is likely to access credit market at lower interest rate by issuing convertibles. Companies with healthy balance sheets are most likely not to issue any types of convertibles as they can easily access cheap credit in the straight debt market. There is no point of sharing future prosperity with an outsider as good and healthy companies can access the credit market with ease. However, the choice of the type of convertible debt a financially distressed company will issue is an open question subject to empirical verifications. Brennan and Schwartz (1988) argue that issuing convertibles can decrease initial interest expense and provides flexibility for future funding change. Mayers (2000) maintains that issuing callable convertible allows for sequential financing need due to potential high growth. Mayers' observation is supported by the issuance of this type of debt by small high growth companies. Information asymmetry between issuer and investor also provides a plausible explanation for issuing convertibles as provisional equity financing. Jalan and Barone-Adesi (1995) study reveals that the convertible as a backdoor equity provides interest tax shield until conversion, which issuing equity does not. Stein (1992) provides empirical evidence that highly rated companies issue straight debt, medium quality firms issue convertible and low credit rated and unrated companies issue equity.

Krishnaswami and Devrim (2008) argue that the choice of convertible over straight debt financing is related to agency cost of debt and is supported by their empirical evidence. Whereas, King and Mauer (2012) provide evidence that the convertible debt call policy is intended to reduce agency cost of debt and equity. To understand the real motive behind issuing straight versus convertible debt we need to look into the type of contingent claims that are associated with issuing debt or equity.

Convertible bonds (CB) may be callable. In this case, this bond is defined as follows:

$CB_c = P + CO_L - CO_S$ 4

where

P is the price of pure bond,

CO_L is a long call granted to investor,

CO_S is a short call granted to the issuer

Callable convertible CB_c can be redefined as

$CB_c = CB - CO_S$ 5

The long call and short call are not offsetting terms as they are extended to investors and issuers respectively with different motivations as to the timing of the exercise of the call options. Therefore, the price dynamics of CB_c is different from the non-callable convertible type due to the short call for the issuer.

MH (1993) extended Dunetz and Mahoney's (1988) duration and convexity of callable bonds to convertible bonds as a portfolio of pure bond and a long call option as follows.

$D^*_{cb} = D \times P/P_{cb} \times 1/(1-\Delta)$ 6

As delta of long call approaches to unity, duration of convertible bond approaches infinity; that means convertible is converted into common stock. Convertible price is always equal to or greater than the price of a pure bond due to the long call granted to the investors. Therefore, the yield on the convertible bond theoretically has to be lower than that of the pure bond. As delta (Δ) of convertible approaches one indicating that a +/- one percent change in the underlying stock leads to a one percentage change in the value of the long call. The change in the value of the callable convertible in equation 5 that is induced by a change in the yield to maturity can be expressed as follows.

$dCB_c/dk = \partial CB/\partial k - (\partial CO_S/\partial CB)(\partial CB/\partial k)$ 7

Rearranging and defining the first derivative of convertible bond with respect to a change in yield is as follows.

$dCB/dK = D_{cb} \times Pcb$ 8

Substituting 6 and 8 into 7 and dividing by the Price of callable convertible bond P_{ccb} and rearranging the terms, we get the duration of callable convertibles as
follows.

$$D^*_{ccb} = D_{cb} \times P_{cb}/P_{ccb} \times (1- \Delta_S) /(1- \Delta_L) \quad 9$$

Where
D^* is duration of straight bond
Δ_S is the delta of the short call and
Δ_L is the delta of the long call.

The timing of the exercise of the calls granted to investors and issuers may differ depending on the motivation of the issuers and investors. However, the issuer can dictate the timing of the conversion by forcing the investor to convert the bond into common stock. In this case, the delta of the long call and short call will be identical.

Equation 9 is the generalized formula for the duration of option embedded bonds. The above derivation in equation (9) at the limit collapses to the formulas derived by Dunetz and Mahoney (1988) for callable bonds as a portfolio of long bond and a short call as well as the duration of the convertible bonds as a portfolio of long bond and long call developed by MH (1993). This derivation is demonstrated below.

Equation 9 reduces to Dunetz and Mahoney (1988) for callable bonds as follows:

$$D^*_{cb} = D \times P/P_{cb} \times (1- \Delta) \quad 10$$

Rearranging terms in equation 3, the second derivative of the bond with respect to a change in yield is as follows.

$$\partial P^2/\partial K^2 = P \times C_p \quad 11$$

Where, P and C_p are respectively, the price and the convexity of the pure bond,

$$\partial P^2_{cb} /\partial K^2 = P_{cb} \times C_{cb} \quad 12$$

MH initially develop the convexity of convertible bonds C_{cb} that is as follows:

$$C_{CB} = 1/(1- \Delta) \times [C_p \times P/P_{cb} + \Gamma \times P_{cb} \times D^2_{cb}] \quad 13$$

Where,
Γ is the gamma of the option and
C_p is the convexity of the pure bond.

The convexity of the convertible bonds is greater than that of a straight bond by a factor that is equal to the value of the second term in the RHS equation 13. Therefore, the yield to maturity of the convertible has to be less than that of the pure bond. When delta is zero, that is, an option is out of the money, the convexity of the convertible bond approaches, the convexity of the pure bond. That is;

$C_{cb} = C_p$

However, at the limit when the delta is equal to unity, the convexity of the convertible approaches the convexity of the underlying stock that is infinite.
The expression in equation 13 can be compared to that of Dunetz and Mahoney for the convexity of the callable bonds as follows.

$$C_{CB} = P_{SB}/P_{CB}\,[C_{SB}\,(1-\Delta) - P_{SB} \times \Gamma \times D^2_{SB}] \qquad 14$$

As can be verified from the above expression, the convexity of the callable bond is less than that of the pure bond by a factor equal to the 2nd marginal term in bracket of equation 14. Therefore, the price of the callable bond is less than that of the straight bond and its yield to maturity has to be greater than that of the pure bond. However, careful examination of the equation 14; that is the convexity of the callable bonds derived by Dunetz and Mahoney reveal an error. For example, when the delta is zero and call is worthless, the convexity of the callable bond approaches that of straight bond in 14 as both delta and gamma are equal to zero. On the other hand, when Delta is equal to 1; that means the bond is called as its duration approaches zero, while convexity will be equal to the marginal term in the bracket on the RHS of the equation 14 ($P_{SB} \times \Gamma \times D^2_{SB}$) as opposed to being equal to zero.

In our derivation of the callable convertible bond's duration and convexity, as delta approaches 1, duration of callable bonds D^*_{cb} approaches zero once the bond is called. However, when the delta is zero, the short call is worthless and duration of callable will be equal to that of straight bond in equation 10. On the other hand, the convexity of the callable bond follows the same behavior when the delta is 1, as shown in Dunetz and Mahoney as well as in MH derivation. In case delta is zero, the convexity of the callable bond approaches zero as well, which is in stark contrast to non-zero convexity in Dunetz and Mahoney. Furthermore, equation 9 reduces to that of MH for convertible bonds as in equation 6 assuming the convertible bond is not callable. The generalized algorithm in equation 9 provides testable hypotheses about various issues. These are regarding the optimum call policy, as well as yield differential between callable and non-callable, convertibles; callable v non-callable denominated in Dollar, Yen, Euro, Pound to name a few, and bond market efficiency in pricing mechanisms where bid/ask spread is fairly substantial, particularly in the emerging markets bond funds. Equation 9 also provides a testable hypothesis on agency related issues between creditors and stockholders and the means in which agency related issues are mitigated through issuing convertibles. The generalized algorithm also provides a convenient way of estimating the dollar value of 1 basis point change in yield known as DV01, an important metric in the bond market.

Convexity of a callable convertible can be derived from the equation 9 as follows;

$$C_{ccb} = (1-\Delta s) \times 1/(1-\Delta_L) \times P/P_{cb}\,[C_p + \Gamma \times P_{cb} \times D^2_{cb}] \qquad 15$$

At the time of the exercise of the short call forcing conversion both delta in equation 15 converges to unity as

$\Delta_L = \Delta_s = \Delta$

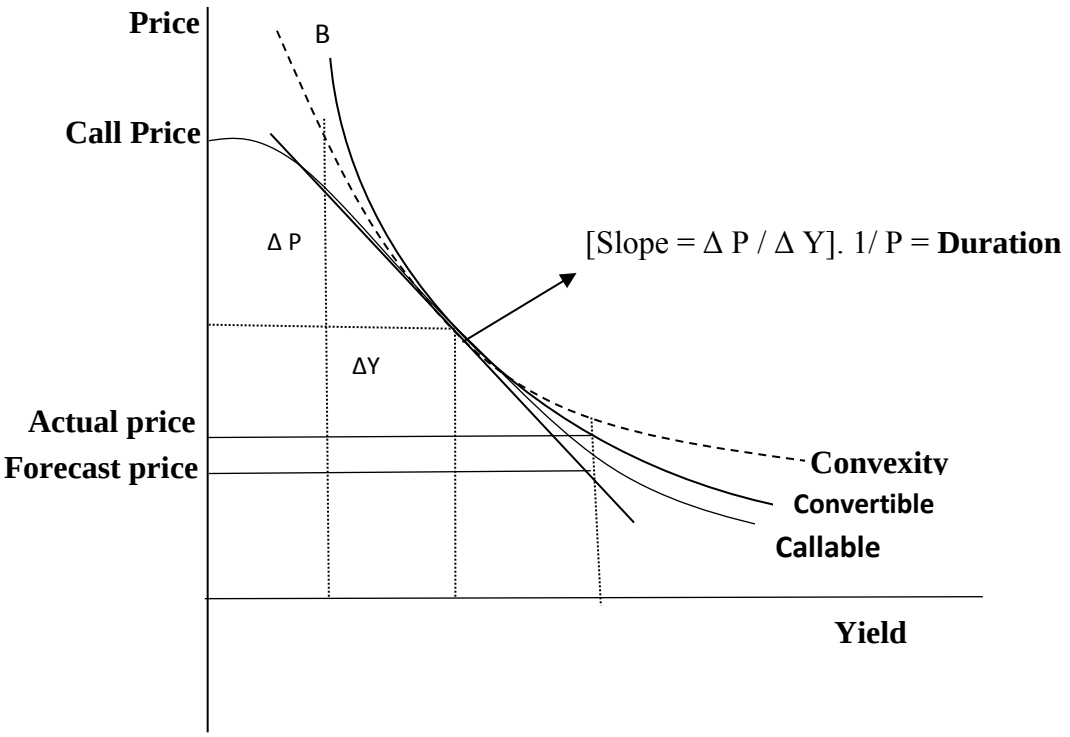
Moreover, the convexity of the callable convertible approaches the convexity of the underlying stock.
We illustrate price dynamic of option-embedded bonds (callable, convertible, etc.) in Figure 1 below. The graph shows that duration and convexity underestimate the actual price change when yield falls and overestimates it when yield rises.

Example: Consider a 6 percent coupon bond rated to yield 6 percent. This bond has a maturity of 15 years and face value of $1000. The coupon is paid once a year. Duration of this bond is 10.21 year and modified duration is 9.63 year. The convexity of the above bond is estimated to be 126.91. Approximating percentage change in the price of a bond using both duration and convexity provide a far more reliable estimate of the actual change. For example, the percentage price change for +/- 100 basis points change in yield produce an expected change in the price of +10.26 percent, or -9 percent. Duration and convexity produce expected percentage change in the price of 10.26 percent versus actual percentage price change of 10.37 percent. Therefore, duration and convexity combined underestimate actual price change for a 100 basis points decrease in yield by 11 basis points. However, duration and convexity combined overestimate actual price change by 10 basis points for a 100 bps increase in yield.

$\Delta P/P = - D_p . \Delta Y + ½ C . (\Delta Y)^2$
$=- 9.63(+/- .01) + ½ \times 126.91 \times (+/-.01)^2$
$= +/- .0963+.0063=.1026$ or $-.09$

Actual price change for +/- 100 basis points change in yield produce
+ 10.37% and – 9.1%

Figure 1. Price Yield Relationship



As can be seen from the graph, convertible bond offering investors upside potential in falling interest rate scenario and downside protection when yield goes up. When the yield rises, a convertible bond falls less than a straight bond, but much less a callable bonds as long call option is directly related to the interest rate as yield increases, the long call value increases as well. While rising yield causes all bonds prices to drop, as there is an inverse relationship between price and yield, the fall in a convertible bond's price is moderated by the rising price of the long call. This is the downside protection afforded to holders of convertible bonds. On the other hand, when yield decreases, the price of convertible bonds increase more than any other bond compliment of the long call. Callable bonds suffer the most in the rising interest rate environment and offer little or no upside potential under falling interest rate scenario as investors try to call the bond and reissue at a lower rate. Home mortgages are the example of the callable bonds; as homeowners are likely to extend the life of mortgage by simply paying the monthly mortgage without paying any prepayment. However, homeowners find it advantag Yield the bond in the falling interest rate environment to take advantage of economic ben associated with calling the old bond and refinancing at a lower rate. Callable convertible bonds are a complicated instrument as there are two competing calls; the long call for the investors and short calls for the issuer. The upside potential of the callable convertible may be limited by the forced conversion initiated by the issuer independent of the interest rate scenario as agency related issues may play a role. Furthermore, forced conversion may be initiated to window dress

the balance sheet for future debt financing due to the improvement in the balance sheet and economic fundamentals of the issuing company.

Implications and Concluding Remarks

We derive a generalized formula for option embedded bonds that relates duration and convexity of the callable, convertible to that of the option-free bonds. In particular, our derivation of a generalized formula for duration and convexity tends to be identical to that of Dunetz and Mahoney (1988) for a callable bond duration as a special case. Furthermore, MH (19992) derivation of duration and convexity for convertible bonds becomes a special case in our generalized formula. Callable convertibles are unique financial instrument having characteristics and price dynamics of stocks and bonds. There is much potential for investigating this hybrid financing and investment vehicle. We find that convexity of the callable bond developed by Dunetz and Mahoney fails the test of parameter restrictions on the value of delta of the call options. As we clearly pointed out, once delta approaches 1, the callable bond is called and its duration and convexity tend to zero as the bond disappears from the balance sheet of the issuing company. Dunetz and Mahoney's convexity remains non-zero when the delta is equal to 1.

Our derivation of convexity of the option embedded bonds approaches to zero for callable bonds and to the convexity of underlying stock once the bond is called. The duration and convexity of the callable convertible bonds developed in this paper provide testable hypotheses about some fundamental issues regarding the exercise of the call options by the investors or by the issuers through forced conversion.

Furthermore, our generalized formula provides plausible testable hypotheses on agency related problems between stockholders and bondholders as well as agency issues between managers (through short calls granted to senior managers as backdoor compensation) and creditors. For example, when delta approaches 1, and the long call is in the money, it pays off for investors to convert the bond into common stock. On the other hand, it pays the issuing company to force conversion through short call when call price of the convertible is equal to conversion value; the product of conversion ratio and stock price. Whether or not the call policy of the company is dictated by economics (optimal) or window dressing (not optimal) can be investigated by finding the magnitude of delta. A delta close to close to 1 may reveal that the call policy of the issuing company is not optimal. However, if delta is found to be statistically different from one, may provide evidence that the call policy of the firm is optimal. Ingersoll (1977) reveals that the median firm in his sample of 179 issues of convertible bonds waited until the conversion value was greater than its call price by 43 percent before forcing conversion. Therefore, leading him to conclude that the firms in his sample did not pursue an optimum call policy. We argue that the firms in Ingersoll (1977) sample were not motivated by economics, but by window dressing, a testable hypothesis in future research.

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Table 1: Distribution of Rated and Not Rated Convertibles

Rated		Not Rates/NA
A+	3	
AA-	1	
A	11	
A-	5	
	20	
BBB+	6	
BBB	12	
BBB-	16	
BB+	8	
BB	12	
BB-	10	
B+	3	
B	2	
B-	4	
	73	
CCC+	2	
CCC	3	
CCC-	1	
CC+	1	
	7	
	100	2900
Total		3000

Source: Bloomberg Financials

Table 2: Types of Convertibles

Maturity Type	
Convertible	1469
Conv/Call	216
Conv/Call/Perp	136
Conv/Perp	344
Conv/Put	561
Conv/Put/Call	257
Conv/Sink	14
Conv/Call Sink	2
Conv/Put/Call/SF	1
	3000