# Peg Solitaire on Graphs: Current Results and Open Problems

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# Description of the game

Peg solitaire is a table game which traditionally begins with “pegs” in every space except for one which is left empty (in other words, a “hole”). If in some row or column two adjacent pegs are next to a hole (as in Figure 1), then the peg in $x$ can jump over the peg in $y$ into the hole in $z$. The peg in $y$ is then removed. The goal is to remove every peg but one. If this is achieved, then the board is considered solved. For more information on traditional peg solitaire, refer to Beasley [1] or Berlekamp et al. [8]

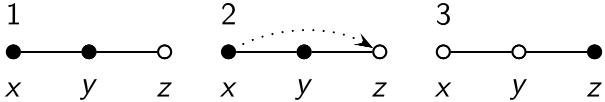


Figure: A Typical Jump in Peg Solitaire

# A Brief History (part 1)

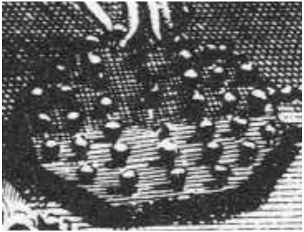

Figure: *Madame la Princesse de Soubise joüant au jeu de Solitaire* by Claude-Auguste Berey, 1697.

# A Brief History (part 2)

Not so very long ago there became widespread an excellent kind of game, called *Solitaire*, where I play on my own, but as with a friend as witness and referee to see that I play correctly. A board is filled with stones set in holes, which are removed in turn, but none (except the first, which may be chosen for removal at will) can be removed unless you are able to jump another stone across it into an adjacent empty place, when it is captured as in Draughts. He who removes all the stones right to the end according to this rule, wins; but he who is compelled to leave more than one stone still on the board, yields the palm.

Gottfried Wilhelm Leibniz,
*Miscellanea Berolinensia* **1** (1710) 24.

# The version you are most likely familiar with...

# The generalization to graphs

In a 2011 paper (B-Hoilman [4]), the game is generalized to graphs in the combinatorial sense. So, if there are pegs in vertices $x$ and $y$ and a hole in $z$, then we allow $x$ to jump over $y$ into $z$ provided that $xy \in E$ and $yz \in E$. The peg in $y$ is then removed.

In particular, we allow 'L'-shaped jumps, which are not allowed in the traditional game.

# Definitions from [4]

- A graph $G$ is *solvable* if there exists some vertex $s$ so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of a single peg.
- A graph $G$ is *freely solvable* if for all vertices $s$ so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of a single peg.
- A graph $G$ is *k-solvable* if there exists some vertex $s$ so that, starting with $S = \{s\}$, there exists an associated minimal terminal state consisting of $k$ nonadjacent pegs.
- In particular, a graph is *distance 2-solvable* if there exists some vertex $s$ so that, starting with $S = \{s\}$, there exists an associated terminal state consisting of two pegs that are distance 2 apart.

# Known results [4]

The usual goal is to determine the necessary and sufficient conditions for the solvability of a family of graphs. To date, the solvability of the following graphs has been determined:

- $K_{1,n}$ is $(n-1)$-solvable; $K_{n,m}$ is freely solvable for $n, m \geq 2$.
- $P_n$ is freely solvable iff $n = 2$; $P_n$ is solvable iff $n$ is even or $n = 3$; $P_n$ is distance 2-solvable in all other cases.
- $C_n$ is freely solvable iff $n$ is even or $n = 3$; $C_n$ is distance 2-solvable in all other cases.
- The Petersen Graph, the platonic solids, the archimedean solids, the complete graph, and the $n$-dimensional hypercube are freely solvable.

# Known Results (Cont)

- The double star $DS(L, R)$ is freely solvable iff $L = R$ and $R \neq 1$; $D(L, R)$ is solvable iff $L \leq R + 1$; $DS(L, R)$ is distance 2-solvable iff $L = R + 2$; $DS(L, R)$ is $(L - R)$-solvable in all other cases [5].
- The solvability of all graphs with seven vertices or less [2].
- If $G$ and $H$ are solvable (or distance 2-solvable), then the Cartesian product $G \Box H$ is solvable [4].

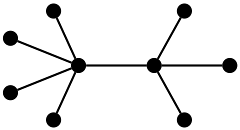

Figure: The Double Star - $DS(4, 3)$

# An Open Problem

We know that even cycles are freely solvable and that odd cycles are distance 2-solvable. What about chorded odd cycles? In other words, cycles $C(n, m)$ on $n$ vertices in which an edge has been added between vertices 0 and $m$.

- For all $n$ and $m \leq n$, the chorded odd cycle $C(2n+1, m)$ is solvable [2].
- For all $n$, $C(n, 2)$ is freely solvable [3].
- If $n \leq 9$ and $m \leq n$, then $C(2n+1, m)$ is freely solvable [2].
- **Open Problem** - Are all chorded odd cycles freely solvable?

# Classification of trees

If $G$ is a (freely) solvable subgraph of $H$, then $H$ is (freely) solvable. Since every connected graph has a spanning subtree, a natural (and very important) problem is to determine which trees are solvable. All trees of diameter three or less were classified in [4, 5]. B-Walvoort took the next natural step by classifying the solvability of trees of diameter four [7].

# Parametrization

Diameter 4 trees will be parameterized as $K_{1,n}(c; a_1, \ldots, a_n)$, where $n$ is the number of non-central support vertices, $c$ is the number of pendants adjacent to $x$ and $a_i$ is the number of pendants adjacent to $y_i$. Without loss of generality, we may assume that $a_1 \geq \ldots \geq a_n \geq 1$.

Also, let $k = c - s + n$, where $s = \sum_{i=1}^{n} a_i$.

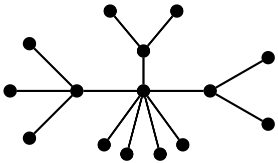

Figure: The graph $K_{1,3}(4; 3, 2, 2)$

# Trees of diameter 4

**Theorem 1 [7]**

Assume $a_1 \geq 2$. The conditions for solvability of such diameter four trees are as follows:

(i) The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is solvable iff $0 \leq k \leq n+1$.

(ii) The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is freely solvable iff $1 \leq k \leq n$.

(iii) The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is distance 2-solvable iff $k \in \{-1, n+2\}$.

(iv) The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is $(1-k)$-solvable if $k \leq -1$. The graph $K_{1,n}(c; a_1, \ldots, a_n)$ is $(k-n)$-solvable if $k \geq n+2$.

# More on trees of diameter 4

**Theorem 2 [7]**

The conditions for solvability of $K_{1,n}(c; 1, ..., 1)$ is as follows:

(i) The graph $K_{1,2r}(c; 1, \ldots, 1)$ is solvable iff $0 \leq c \leq 2r$ and $(r, c) \neq (1, 0)$. The graph $K_{1,2r+1}(c; 1, \ldots, 1)$ is solvable iff $0 \leq c \leq 2r + 2$.

(ii) The graph $K_{1,n}(c; 1, \ldots, 1)$ is freely solvable iff $1 \leq c \leq n - 1$.

(iii) The graph $K_{1,2r}(c; 1, \ldots, 1)$ is distance 2-solvable iff $c = 2r + 1$ or $(r, c) = (1, 0)$. The graph $K_{1,2r+1}(c; 1, \ldots, 1)$ is distance 2-solvable iff $c = 2r + 3$.

(iv) The graph $K_{1,2r}(c; 1, \ldots, 1)$ is $(c - 2r + 1)$-solvable if $c \geq 2r + 1$. The graph $K_{1,2r+1}(c; 1, \ldots, 1)$ is $(c - 2r - 1)$-solvable if $c \geq 2r + 3$.

# Open Questions Concerning Trees

Classification of solvable trees is one of the most important open problems in my opinion. In particular, what are the necessary and sufficient conditions for the solvability of diameter five trees, caterpillars, and asters?

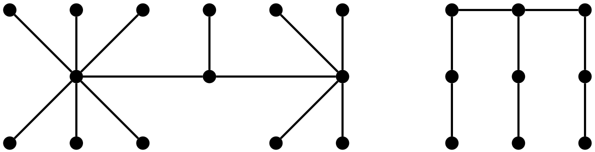

Figure: A Caterpillar and an Aster

# Open Questions Concerning Trees (Cont)

Another interesting question involves the percentage of trees that are solvable. Suppose that $T_n$ is the number of non-isomorphic trees on $n$ vertices and that $S_n$ is the number of solvable non-isomorphic trees on $n$ vertices. In particular, what are the values of $S_n/T_n$? Does the sequence have a limit?

Knowledge of the sequence $S_n/T_n$ would give insight into the number of solvable graphs. It would also provide a partial bound to the extremal problem, namely what is the maximum number of edges in an unsolvable connected graph (see Aaron Gray's talk).

# Freely solvable graphs

It seems to be the case that most graphs are freely solvable. In fact, of the 996 connected non-isomorphic graphs on seven vertices or less, only 54 are not freely solvable. However, showing that a given graph (or family of graphs) is freely solvable often proves difficult.

In addition to the above results, B-Gray [3] showed that meshes (i.e., $P_n \square P_m$) and "fat triangles" are freely solvable.

If for all $s, t \in V$, there exists a peg solitaire solution of $G$ with initial hole in $s$ and final peg in $t$, then we say that $G$ is *totally solvable*. It is an open problem as to which graphs are totally solvable.

# Fool's Solitaire

In *fool's solitaire*, the player tries to leave the maximum number of pegs possible under the caveat that the player jumps whenever possible. This maximum number will be denoted $Fs(G)$.

If $G$ is a connected graph, then a sharp upper bound for the fool's solitaire number is $Fs(G) \leq \alpha(G)$, where $\alpha(G)$ denotes the independence number of $G$ [6].

# Known Results from [6]

B-Rodriguez determined the fool's solitaire number for several families of graphs. In particular:

- $Fs(K_{1,n}) = n$.
- $Fs(K_{n,m}) = n - 1$ if $n \geq m \geq 2$.
- $Fs(P_n) = \lfloor n/2 \rfloor$.
- $Fs(C_n) = \lfloor \frac{n-1}{2} \rfloor$.
- $Fs(Q_n) = 2^{n-1} - 1$.
- The fool's solitaire number for all connected graphs with six vertices or less.

# A natural conjecture

In all of the above cases, $Fs(G) \geq \alpha(G) - 1$. In fact, of the 143 connected graphs with six vertices or less, 130 satisfy $Fs(G) = \alpha(G)$. So a natural conjecture is that

$$\alpha(G) - 1 \leq Fs(G) \leq \alpha(G).$$

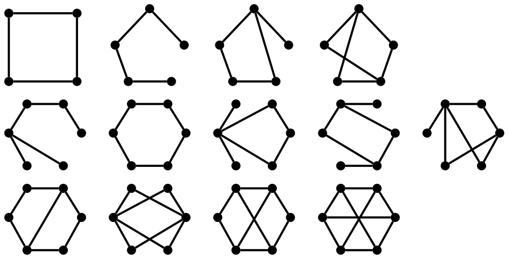

Figure: Graphs with $n(G) \leq 6$ such that $Fs(G) = \alpha(G) - 1$

# A counterexample...

However, trees of diameter four provide an infinite class of counterexamples to the above conjecture.

**Theorem 3 [7]**

Consider the diameter four tree $G = K_{1,n}(c; a_1, \ldots, a_n)$, where $a_i \geq 2$ for $1 \leq i \leq n - \ell$, $a_i = 1$ for $n - \ell + 1 \leq i \leq n$, and $n \geq 2$.

(i) If $c = 0$ and $\ell = 0$, then $Fs(G) = s + c - \left\lfloor \frac{n}{3} \right\rfloor$.

(ii) If $c \geq 1$ and $\ell = 0$, then $Fs(G) = s + c - \left\lfloor \frac{n+1}{3} \right\rfloor$.

(iii) If $\ell \geq 1$, then $Fs(G) = s + c - \left\lfloor \frac{n-2m+1}{3} \right\rfloor$, where $m = \min\{\ell, \left\lfloor \frac{n}{2} \right\rfloor\}$.

Note that the difference between $\alpha(G)$ and $Fs(G)$ can be arbitrarily large in trees of diameter four. However, $Fs(G) > 5\alpha(G)/6$ for all such trees.

# Open Problems for fool's solitaire

- What other graphs have $Fs(G) < \alpha(G) - 1$?
- Is there a non-trivial lower bound on $Fs(G)$?
- For what graphs does edge deletion lower the fool's solitaire number?
- How much can edge deletion lower the fool's solitaire number?

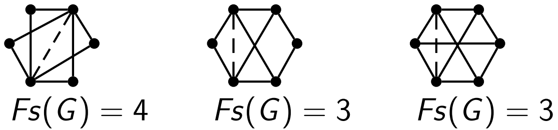


Figure: Graphs in which Edge Deletion Lowers $Fs(G)$

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