The Extremal Problem in Peg Solitaire on Graphs

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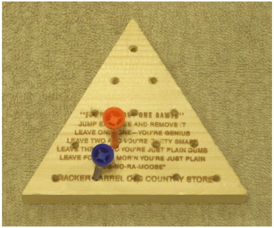
Twenty-Sixth Cumberland Conference on
Combinatorics, Graph Theory, and Computing
May 24, 2013

Peg Solitaire

Peg solitaire is played on a board with numerous *holes*. *Pegs* are placed in every hole but one. A peg is removed by *jumping* over it with an adjacent peg into an adjacent hole. The game ends when no further moves are possible. The board is solved if only one peg remains. See Beasley [1] or Berlekamp et al [9] for more info.

Figure: A Typical Jump in Peg Solitaire

1 

2 

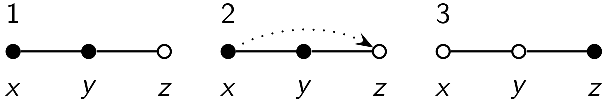
Famous Examples

Peg Solitaire on Graphs

- Beeler and Hoilman [6] generalized the game to arbitrary boards, which are treated as graphs in the combinatorial sense.
- We assume all graphs are finite, undirected, graphs with no loops or multiple edges.
  Particularly, we assume all graphs are connected.
- If there are pegs in vertices x and y and a hole in z, then x may jump over y into z provided that $xy, yz \in E$. The peg in y is then removed. We denote this jump with $x \cdot \overrightarrow{y} \cdot z$.

Figure: A Typical Jump in Peg Solitaire on Graphs



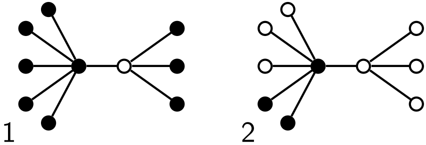
Solution States

- Graph G is *solvable* if there exists some vertex $s \in G$ such that, starting with a hole in s, there exists an associated terminal state consisting of a single peg.
- Graph G is *freely solvable* if for all vertices $s \in G$, starting with a hole in s, there exists an associated terminal state consisting of a single peg.
- Graph G is *k-solvable* if there exists some vertex $s \in G$ such that, starting with a hole in s, there exists an associated terminal state consisting of k nonadjacent pegs.
- Graph G is *distance 2-solvable* if there exists some vertex $s \in G$ such that, starting with a hole in s, there exists an associated terminal state consisting of two pegs that are distance 2 apart.

Useful Results

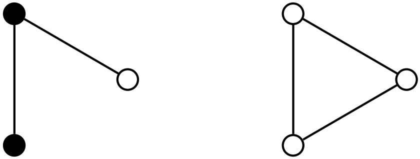
- The complete graph is freely solvable [6].
- The double star $DS(L, R)$ is: (i) freely solvable iff $L = R$ and $R \neq 1$ (ii) solvable iff $L \leq R + 1$ (iii) distance 2-solvable iff $L = R + 2$ (iv) $(L - R)$-solvable if $L \geq R + 3$ [7].
- Double stars may be used to quickly eliminate pegs in an action called a *purge* [2].

Figure: The Double Star Purge on $DS(5, 3)$



P_3

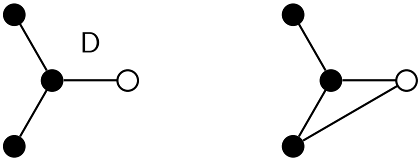
The path on three vertices is solvable, at best, by [6]



However, the addition of an edge results in the complete graph on three vertices, which is freely solvable by [6].

$K_{1,3}$

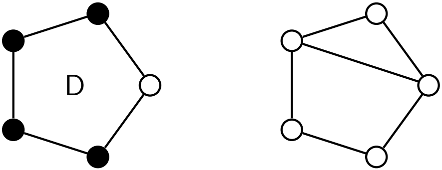
The star on four vertices is distance-2 solvable by [6].



However, the addition of an edge results in a graph that is solvable by [4].

C_5

The cycle on five vertices is distance-2 solvable by [6].



However, the addition of an edge results in a chorded odd cycle. Chorded odd cycles $C(2n+1,2)$ are freely solvable by [3].

Edge Addition

Edge addition may...

- ...make an unsolvable graph solvable or even freely solvable.
- ...make a solvable graph freely solvable.

This seems to make sense, since the addition of an edge results in a greater number of possible jumps.

Solvability Chart

Graphs on Seven Vertices					
Edges	6	7	8	9	10
Percent Solvable	54.5%	87.9%	98.5%	100%	100%

After nine edges, all graphs of order seven are at least solvable.

Graphs on Seven Vertices					
Edges	6	7	8	9	10
Percent Freely Solvable	0%	53.1%	92.3%	98.1%	100%

After 10 edges, all graphs of order seven are freely solvable.

More Results

- All graphs on 4 vertices and 4 (5) edges are solvable (freely solvable).
- All graphs on 5 vertices and 6 (7) edges are solvable (freely solvable).
- All graphs on 6 vertices and 7 (8) edges are solvable (freely solvable).
- All graphs on 7 vertices and 9 (10) edges are solvable (freely solvable).

The Extremal Problem

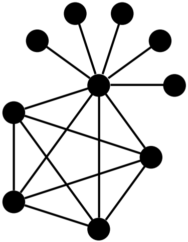
- What is the maximum number of edges in an unsolvable graph on n vertices?
- We denote this number $\tau(n)$.
- A graph G is *edge k-critical* if G is k-solvable, but the addition of any edge reduces the number of pegs at the end of the game.
- In particular, G is *edge critical* if G is not solvable, but the addition of any edge results in a solvable graph.
- We call an unsolvable (solvable but not freely solvable) graph G a *critical graph* if the addition of any edge to G results in a solvable (freely solvable) graph.

Examined

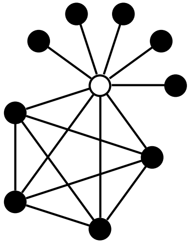
Theorem

The generalization of the complete graph and the star, $K_n(n, 0, ..., 0)$, is a critical graph [2].

Figure: $K_5(5, 0, 0, 0, 0)$



Proof Overview



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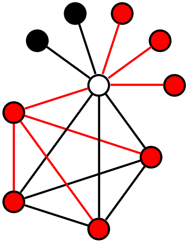

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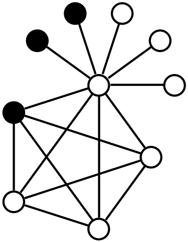

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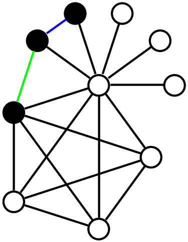

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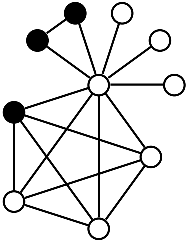

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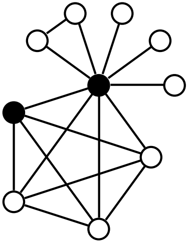

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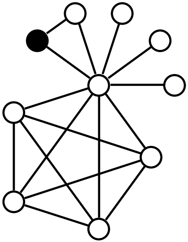

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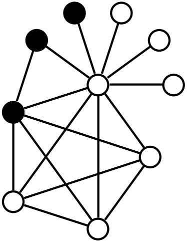

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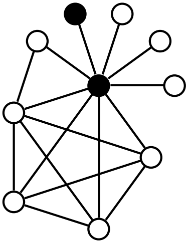

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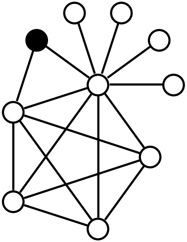

Proof Overview

The remainder of the proof follows similarly.

A similar argument is used to show that $K_n(n+1, 0, ..., 0)$ is also a critical graph.

Current Bounds [2]

- If n is odd, say $n = 2k + 1$, then $\tau(2k + 1) \geq \frac{k(k+1)}{2} + 1$.
- If n is even, say $n = 2k$, then $\tau(2k) \geq \frac{k(k+1)}{2}$.

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- These bounds are sharp.
- Trivially, $\tau \leq \frac{n(n-1)}{2}$.
- This current bound is approximately $\frac{n^2}{8}$.
- Conjecture (Beeler): $\tau(2k) = \frac{k(k+1)}{2}$.

Thanks

Thank you for attending.

May all your graphs be freely solvable!

References

[1] John D. Beasley. *The ins and outs of peg solitaire*, volume 2 of *Recreations in Mathematics*. Oxford University Press, Eynsham, 1985.
[2] Robert A. Beeler and Aaron D. Gray. Extremal results for peg solitaire on graphs. In preparation.
[3] Robert A. Beeler and Aaron D. Gray. Freely solvable graphs in peg solitaire. In preparation. [4] Robert A. Beeler and Aaron D. Gray. Peg solitaire on graphs with seven vertices or less. *Congressus Numerantium*. Volume 211, pp. 151-159, 2012.
[5] Robert A. Beeler, Aaron D. Gray, and D. Paul Hoilman. Constructing solvable graphs in peg solitaire. *Bulletin of the ICA*. Volume 66, pp. 89-96, 2012.
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[7] Robert A. Beeler and D. Paul Hoilman. Peg solitaire on the windmill and the double star graphs. *The Australasian Journal of Combinatorics*, 52:127-134, 2012.
[8] Robert A. Beeler and Tony K. Rodriguez. Fool's Solitaire on Graphs. To Appear in *Involve*.
[9] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning ways for your mathematical plays. Vol. 2*. A K Peters Ltd., Natick, MA, second edition, 2003.
[10] Pictures from the internet and Aaron D. Gray.