Exact minimum d-degree thresholds for Hamilton cycles in k-uniform Hypergraphs

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Joint work with Yi Zhao

Outline

1. Introduction and Main results
2. Proof Ideas
3. Concluding Remarks and Open Problems

Warm up

- A k-uniform hypergraph (k-graph) H on V: $H \subset \binom{V}{k}$.
- Minimum d-degree: $\delta_d(H) = \min_{S \in \binom{V}{d}} \{\#$ of edges containing $S\}$
- An k-uniform ℓ-cycle is a k-graph which admits a cyclic ordering of the vertices such that each edge contains k consecutive vertices and two consecutive edges share ℓ vertices.
- Tight cycles: $\ell = k - 1$; Loose cycles: $\ell = 1$.
- Dirac '52: *every graph G of order $n \geq 3$ with min-degree $\delta(G) \geq n/2$ contains a Hamilton cycle*.

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Dirac Type Results in Hypergraphs

- (Katona-Kierstead 99) $\delta_{k-1}(H) \geq (1 - \frac{1}{2k})n + 4 - k - \frac{5}{2k} \Rightarrow$ a tight H-cycle.
- (Rödl-Ruciński-Szemerédi 08) $\delta_{k-1}(H) \geq (\frac{1}{2} + o(1))n \Rightarrow$ a tight H-cycle.
- (RRS 04, 11) For $k = 3$, $\delta_2(H) \geq \lfloor \frac{n}{2} \rfloor \Rightarrow$ a tight H-cycle.
- (Kühn-Osthus 06) $k = 3$, $\delta_2(H) \geq (\frac{1}{4} + o(1))n \Rightarrow$ a loose H-cycle.
- (Keevash-K-Mycroft-O 10) $\delta_{k-1}(H) \geq (\frac{1}{2(k-1)} + o(1))n \Rightarrow$ a loose H-cycle.
- (Hàn-Schacht 10) For $0 < \ell < k/2$, $\delta_{k-1}(H) \geq (\frac{1}{2(k-\ell)} + o(1))n \Rightarrow$ a H ℓ-cycle.

Dirac Type Results in Hypergraphs (continued)

1. (KMO 10) For $0 < \ell < k$ such that $k - \ell \nmid k$,

$$\delta_{k-1}(H) \geq \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + o(1) \right) n$$

$\Rightarrow$ a H ℓ-cycle.

2. (Buss-H-S 12) For $k = 3$, $\delta_1(H) \geq (\frac{7}{16} + o(1))\binom{n}{2} \Rightarrow$ a loose H-cycle.

Main result 1

Theorem 1 (H. Yi Zhao 13+)

$\exists n_0$ such that the following holds. Suppose that H is a 3-graph on $n > n_0$ with $n \in 2\mathbb{N}$ and

$$\delta_1(H) \geq \binom{n-1}{2} - \binom{\lfloor \frac{3}{4}n \rfloor}{2} + c,$$

where $c = 2$ if $4 \mid n$, $c = 1$ if $4 \nmid n$. Then H contains a loose H-cycle.

Main result 2

Theorem 2 (H. Yi Zhao 13+)

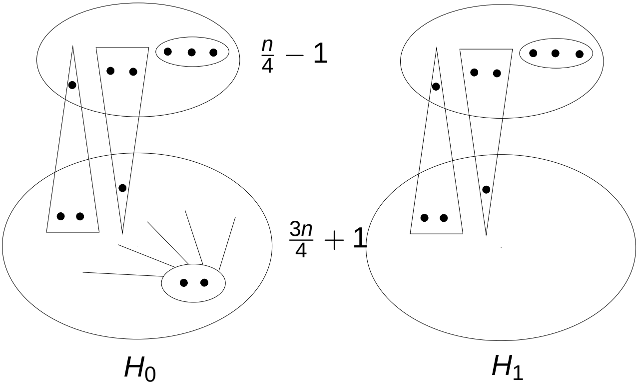
For $k \geq 3$ and $0 < \ell < k$ such that $k - \ell \nmid k$, $\exists n_0$ such that the following holds. Suppose that H is a k-graph on $n > n_0$ with $n \in (k-\ell)\mathbb{N}$ and

$$\delta_{k-1}(H) \geq \frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} n,$$

Then H contains a H ℓ-cycle.

Lower Bound Construction

The following constructions show that Theorem 1 and 2 are best possible ($k = 3$).



Absorbing Lemma

Lemma 3 (Absorbing Lemma, B-H-S, 12)

$\forall \gamma > 0$, $\exists n_0$ *such that:* $\forall$ *3-graph* H *on* $n > n_0$ *vertices with* $\delta_1(H) \geq \frac{13}{32}\binom{n}{2}$, $\exists$ *a loose path* $\mathcal{P}$ *with* $|V(\mathcal{P})| \leq \gamma n$ *such that* $\forall U \subset V \setminus V(\mathcal{P})$ *of size* $\leq \gamma^3 n$ *and* $|U| \in 2\mathbb{N}$, $\exists$ *a loose path* $\mathcal{Q}$ *with* $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ *and* $\mathcal{P}$ *and* $\mathcal{Q}$ *have exactly the same ends.*

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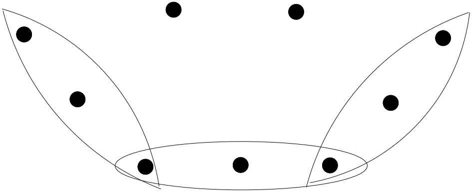

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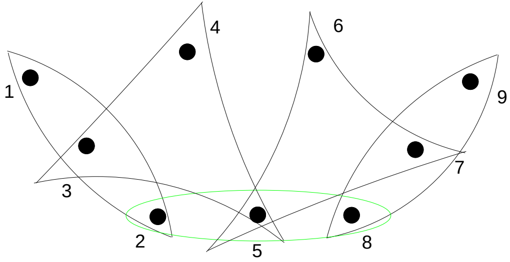

Reservoir lemma

Lemma 4 (Reservoir lemma, B-H-S, 12)

$\forall 1/4 > \gamma > 0$, $\exists n_0$ such that: $\forall$ 3-graph H on $n > n_0$ vertices with $\delta_1(H) \geq (1/4 + \gamma)\binom{n}{2}$, $\exists R \subset V(H)$ with $|R| \leq \gamma n$ and: $\forall (a_i, b_i)_{i \in [k]}$ consisting of $k \leq \gamma^3 n/12$ mutually disjoint pairs of vertices, $\exists \{u_i, v_i, w_i\}_{i \in [k]}$ connecting $(a_i, b_i)_{i \in [k]}$ which contains vertices from R only.

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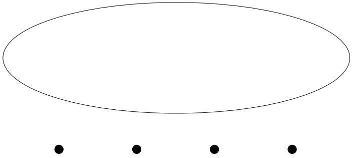

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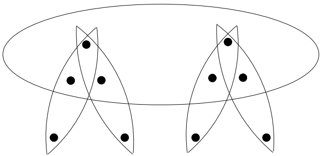

Outline for the proof of the asymptotic result

Buss-Hàn-Schacht: $\delta_1(H) \geq \frac{7}{16}\binom{n}{2} + o(n^2) \Rightarrow$ loose H. cycle.

1. Apply the Absorbing Lemma to find an *reasonably long* absorbing path $P = v_1 \dots v_p$.
2. Apply the Reservoir Lemma to find a *smaller* reservoir set R in $(V \setminus V(P)) \cup \{v_1, v_p\}$.
3. Cover the most vertices of $V \setminus (V(P) \cup R)$ with *constant many* vertex-disjoint loose paths $\{P_i\}$.
4. Connect all the paths $\{P_i\}$ and P by using the vertices of R.
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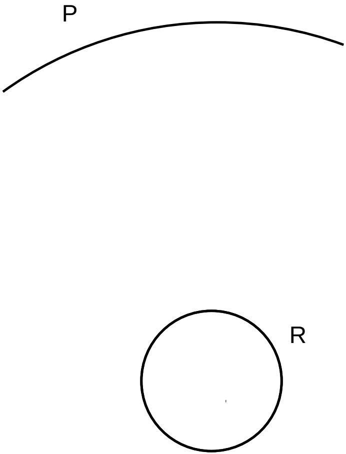

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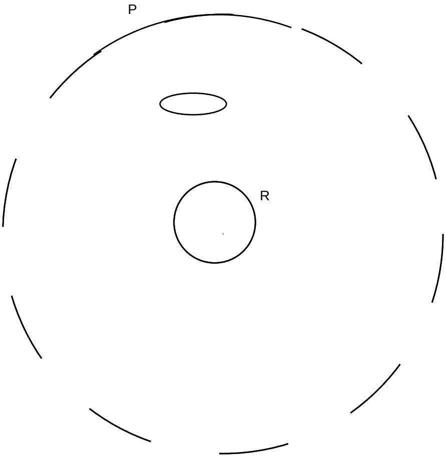

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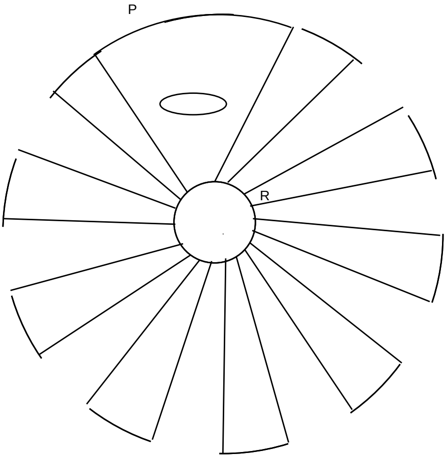

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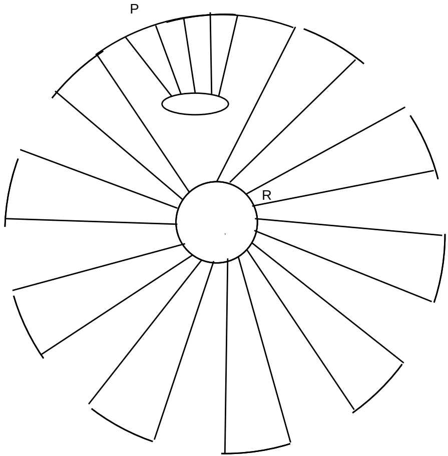

New ingredients

- Separate extremal and non-extremal cases: H is Δ-extremal if $\exists B \subset V$ of size $\lfloor \frac{3n}{4} \rfloor$ s.t. $e(B) \le \Delta n^3$.
- Prove a stronger Path Tiling Lemma: $\forall \gamma, \alpha > 0$, $\exists p \in \mathbb{N}$, $\Delta > 0$ s.t. if $\delta_1(H) \ge (\frac{7}{16} - \gamma)\binom{n}{2}$, then all but at most αn vertices of H can be covered by at most p vertex-disjoint loose paths, unless H is Δ-extremal.

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Tools for the proof of the Path Tiling Lemma

- Fact: a loose path is 3-partite on parts A, B, C s.t. $|A| : |B| : |C| \approx 1 : 2 : 1$.
  1 2 3 2 1 2 3 2 1 2 3 2 1 2 3$\cdots$
- 3-graph Y on $\{w, x, y, z\}$ with edges wxy, xyz.
- Lemma: if a 3-partite 3-graph with 2 parts of size m and one part of size $2m$ is ϵ-regular, then there is one loose path covering the most of its vertices.

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How to prove the Path Tiling Lemma

1. Apply the weak Regularity Lemma to H and obtain the cluster 3-graph H'.
2. Show that H' contains an almost Y-tiling unless H' is extremal.
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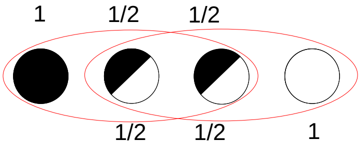

Proof Ideas of Y-tiling

For a maximal Y-tiling $\{Y_1, Y_2, \cdots, Y_m\}$ in K, let V' be set of vertices covered by some copy of Y and $U = V \setminus V'$. Assume that $|U| \geq 2^{20}$. Goal: find a sparse set of size $\lfloor \frac{3}{4}n \rfloor$.

1. $e(U) \leq \frac{1}{3}\binom{|U|}{2}$.
2. $e(UUV') \leq (1 + o(1))m\binom{|U|}{2}$.
3. Almost all systems $\{Y_i, Y_j, u\}$ are stable – as shown in the figure (edges stand for the link graph of u). In this case we say that v covers $\{Y_j, u\}$.
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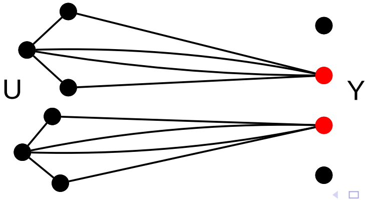

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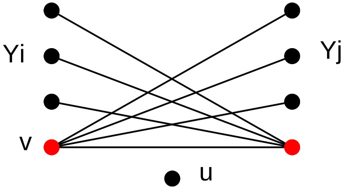

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Concluding Remarks and Open Problems

Remarks:

- This proof may be extended to finding the $(k-2)$-degree threshold of loose Hamilton cycles in k-graphs.
- The proof for our other theorem is similar.

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- Determine the codegree thresholds exactly when $k-\ell$ divides k.
- Other cases, e.g., finding the vertex-degree threshold for tight H. cycles in 3-graphs.

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