26th Cumberland Conference on
Combinatorics, Graph Theory & Computing

May 24 - May 26, 2013
Middle Tennessee State University

The spectral lower bounds for the circumferences of graphs

Rao Li
Dept. of mathematical sciences
University of South Carolina Aiken
Aiken, SC 29801

-G is a simple graph of order n.

-c(G) denotes the circumference, i.e., the length of the longest cycle, in G.

-A cycle C is called a dominating cycle if the order of each component in G[V(G) – V(C)] is less than 2.

-$\sigma_k(G) := \min \{d(x_1) + d(x_2) + \ldots + d(x_k) : \text{where } \{x_1, x_2, \ldots, x_k\} \text{ is an independent set in } G\}$.

-$e_G(S, T) = |\{st : s \in S, t \in T, st \in E(G), \text{ and } S \cap T = \varnothing\}|$

-A graph G is 1 – tough if $r(G - S) \leq |S|$ for every subset S of V(G) with $r(G - S) > 1$, where $r(G - S)$ is the number of components in the graph $G[V(G) - S]$.

-A(G) is the adjacency matrix of G.

-The Laplacian of a graph G is defined as $L(G) = D(G) – A(G)$, where $D(G)$ is the diagonal matrix of the vertex degrees of G.

-The Laplacian eigenvalues $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G) = 0$ of a graph G are the eigenvalues of $L(G)$.

-The signless Laplacian of a graph G is defined as $L^+(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of the vertex degrees of G.

-The eigenvalues $q_1(G) \geq q_2(G) \geq \ldots \geq q_n(G)$ of $L^+(G)$ are called signless Laplacian eigenvalues of G.

Theorem 1 Let G be a graph of order n. If u is vertex in G, then

$$\lambda_{i+1}(G) - 1 \leq \lambda_i(G-u) \leq \lambda_i(G), where \ \ i = 1, 2, \cdots, (n-1).$$

Z. Lotker, Note on deleting a vertex and weak interlacing of the Laplacian spectrum, *Electronic Journal of Linear Algebra* **16** (2007) 68 – 72.

Theorem 2 Let G be a graph of order n. If u is vertex in G, then

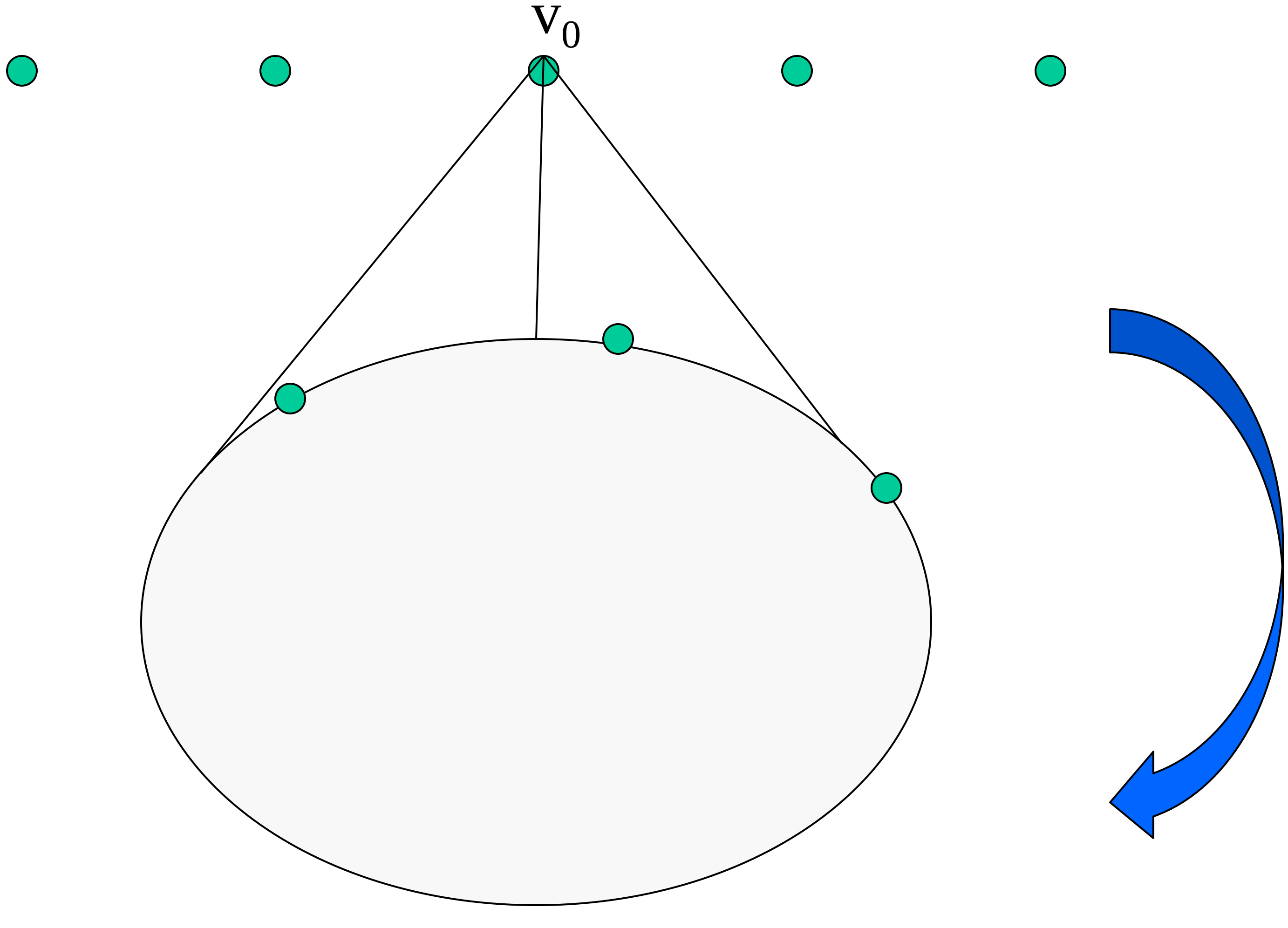
$$q_{i+1}(G) - 1 \leq q_i(G - u) \leq q_i(G), \textit{where} \ \ i = 1, 2, \cdots, (n-1).$$

J. Wang and F. Belardo, A note on the signless Laplacian eigenvalues of graphs, *Linear Algebra and Its Applications* **435** (2011) 2585 – 2590.

The following Theorem 3 is Lemma 8 in

D. Bauer, H. J. Veldman, A. Morgana, E. F. Schmeichel, Long cycles in graphs with large degree sums, *Discrete Math.* **79** (1989/90) 59 – 70.

Theorem 3 Let G be a graph of order n such that $\delta \geq 2$ and $\sigma_3 \geq n$. Let G contain a longest cycle C which is a dominating cycle. If $v_0 \in V(G) - V(C)$ and $A = N(v_0)$, then $(V(G) - V(C)) \cup A^+$ is an independent set of vertices.



v_0

The following Theorem 4 is from Theorem 7 and proof of Theorem 10 in

D. Bauer, H. J. Veldman, A. Morgana, E. F. Schmeichel, Long cycles in graphs with large degree sums, *Discrete Math.* **79** (1989/90) 59 – 70.

Theorem 4 Let G be a 2 – connected graph of order n such that $\sigma_3 \geq n+2$. Then every longest cycle C in G is a dominating cycle and $\max\{\, d(v) : v \in V(G) - V(C)\,\} \geq \frac{\sigma_3}{3}$.

The following Theorem 5 is from Theorem 5 and proof of Theorem 9 in

D. Bauer, H. J. Veldman, A. Morgana, E. F. Schmeichel, Long cycles in graphs with large degree sums, *Discrete Math.* **79** (1989/90) 59 – 70.

Theorem 5 Let G be a 1 – tough graph of order n such that $\sigma_3 \geq n$. Then every longest cycle C in G is a dominating cycle and $\max\{\, d(v) : v \in V(G) - V(C)\,\} \geq \frac{\sigma_3}{3}$.

Theorem 6 Let G be a $2-$connected graph of order n such that $\sigma_3 \geq n+2$. Then $c(G) \geq \min\{\, n, \lambda_{\lfloor \frac{2n-2}{3} \rfloor}(G) + \lceil \frac{n+2}{3} \rceil \,\}$.

Proof of Theorem 6. Let G be a graph satisfying the conditions in Theorem 6. If $c(G) = n$, then the proof is finished. Now we assume that $c(G) < n$. Then Theorem 3 and Theorem 4 imply that there exists a longest cycle C, which is also a dominating cycle, in G such that $d(x) \geq \frac{\sigma_3}{3} \geq \frac{n+2}{3}$, where $x \in V(G) - V(C)$ and for any vertex $w \in V(G) - V(C)$, $d(x) \geq d(w)$. Let S be the set $N(x) \cap V(C)$. Set $N_1 := (V(G) - V(C)) \cup S^+$. Then by Theorem 3 we have that N_1 is independent. Let $N_2 = V(G) - N_1$. Then $|N_2| = n - |N_1| = c - d(x) \leq n - 1 - \frac{n+2}{3} = \frac{2n-5}{3}$. Set $N_2 = \{\, x_1, x_2, ..., x_k \,\}$, where $k = |N_2|$. By Theorem 1, we have

$$\lambda_k(G - x_1) \geq \lambda_{k+1}(G) - 1,$$

$$\lambda_{k-1}(G - x_1 - x_2) \geq \lambda_k(G - x_1) - 1,$$

$$\lambda_{k-2}(G - x_1 - x_2 - x_3) \geq \lambda_{k-1}(G - x_1 - x_2) - 1,$$

....

$$\lambda_{k-(k-1)}(G - x_1 - x_2 - ... - x_k) \geq \lambda_{k-(k-2)}(G - x_1 - x_2 - ... - x_{k-1}) - 1.$$

Summing up the inequalities above, we have

$$\lambda_1(G - x_1 - x_2 - x_3 - ... - x_k) \geq \lambda_{k+1}(G) - k = \lambda_{k+1}(G) - (c - d(x)).$$

Since there is no edge in the graph $(G - x_1 - x_2 - x_3 - ... - x_k)$, $\lambda_1(G - x_1 - x_2 - x_3... - x_k) = 0$. Thus $c \geq \lambda_{k+1}(G) + d(x)$.

Since $k+1 \leq \frac{2n-5}{3}+1 = \frac{2n-2}{3}$, $\lambda_{k+1}(G) \geq \lambda_{\lfloor \frac{2n-2}{3} \rfloor}(G)$. Hence

$$c \geq \lambda_{k+1}(G) + d(x) \geq \lambda_{\lfloor \frac{2n-2}{3} \rfloor} + d(x) \geq \lambda_{\lfloor \frac{2n-2}{3} \rfloor}(G) + \lceil \frac{n+2}{3} \rceil.$$

Thus we complete the proof of Theorem 6. □

Notice that the lower bound in Theorem 6 is attainable for some graphs. For instance, let G be $K_{s,t}$, where $3 \leq s \leq t = 2s-2$. Then G is $2-$connected, $\sigma_3 = n+2 = 3s$, $c(G) = 2s$. Also

$$\lambda_{\lfloor \frac{2n-2}{3} \rfloor}(G) + \lceil \frac{n+2}{3} \rceil = \lambda_{2s-2}(G) + s = s + s.$$

Therefore $c(G) = \min\{\, n, \lambda_{\lfloor \frac{2n-2}{3} \rfloor}(G) + \lceil \frac{n+2}{3} \rceil \,\} = 2\text{s}$.

Using Theorem 2 and arguments similar to those in the proof of Theorem 6, we can prove the following theorem.

Theorem 7 Let G be a 2 – connected graph of order n such that $\sigma_3 \geq n+2$. Then $c(G) \geq \min\{ n, q_{\lfloor \frac{2n-2}{3} \rfloor}(G) + \lceil \frac{n+2}{3} \rceil \}$.

Again, the bipartite graphs $K_{s,t}$, where $3 \leq s \leq t = 2s-2$, show that the lower bound in Theorem 7 is attainable for some graphs.

Using Theorem 1, Theorem 3, Theorem 5, and arguments similar to those in the proof of Theorem 6, we can prove the following theorem.

Theorem 8 Let G be a $1-$tough graph of order n such that $\sigma_3 \geq n$. Then $c(G) \geq \min\{\, n, \lambda_{\lfloor \frac{2n}{3} \rfloor}(G) + \lceil \frac{n}{3} \rceil \,\}$.

Using Theorem 2, Theorem 3, Theorem 5, and arguments similar to those in the proof of Theorem 6, we can prove the following theorem.

Theorem 9 Let G be a $1-$tough graph of order n such that $\sigma_3 \geq n$. Then $c(G) \geq \min\{\, n, q_{\lfloor \frac{2n}{3} \rfloor}(G) + \lceil \frac{n}{3} \rceil \,\}$.

Theorem 10 Let G be a 2 – connected graph of order n such that $\sigma_3 \geq n + 2$. Then $c(G) \geq \min \{n, \sigma_3(n/\lambda_1 + 1)/3\}$.

Proof of Theorem 10 Let G be a graph satisfying the conditions in Theorem 10. If $c(G) = n$, then the proof is finished. Now we assume that $c(G) < n$. Then Theorem 3 and Theorem 4 imply that there exists a longest cycle C, which is also a dominating cycle, in G such that $d(x) \geq \sigma_3/3$, where x is in V(G) – V(C) and for any vertex w in V(G) – V(C),

d(x) ≥ d(w). Let A be the set $N(x) \cap V(C)$.

Set $N_1 := (V(G) - V(C)) \cup A^+$. Then by Theorem 3 we have that

N_1 is independent. Moreover, $|N_1| = n - |C| + d(x) \geq 3$ since

$n - |C| \geq 1$ and $d(x) \geq 2$. Therefore

$$\frac{\sigma_{|N_1|}}{|N_1|} \geq \frac{\sigma_3}{3}.$$

Define $N_2 := V(G) – N_1 = |C| - d(x)$. Then N_1 and N_2 form a partition of V(G). Thus

$$\lambda_1 \geq \frac{e(N_1, N_2)n}{|N_1||N_2|}.$$

Therefore

$$\lambda_1 \geq \frac{e(N_1, N_2)}{|N_1|}\frac{n}{|N_2|} \geq \frac{\sigma_{N_1}}{|N_1|}\frac{n}{(|C| - d(x))}.$$

Solving the inequality, we have that

$$c(G) \geq |C| \geq \frac{\sigma_{N_1}}{|N_1|}(n/\lambda_1) + d(x) \geq \sigma_3(n/\lambda_1 + 1)/3.$$

Hence we complete the proof of Theorem 10.

Notice that the lower bound for $c(G)$ in Theorem 10 is attainable for some graphs. For instance, let G be $K_{p, q}$, where $3 \leq p \leq q = 2p - 2$. Then G is 2 - connected, $\sigma_3 = n + 2 = 3p$, $\lambda_1 = p + q = n$, $c(G) = 2p$, and $\min\{n, \sigma_3(n/\lambda_1 + 1)/3\} = 2\sigma_3/3 = 2p$.

Similarly, we can prove the following theorem.

Theorem 11 Let G be a 1 - tough graph of order n such that $\sigma_3 \geq n$. Then $c(G) \geq \min\{n, \sigma_3(n/\lambda_1 + 1)/3\}$.

Thanks