# WELL-COVERED QUADRANGULATIONS


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Let $\alpha(G)$ denote the independence number of $G$; i.e., the size of a largest independent set of vertices.

Independent set problems are hard!!!

And no wonder!

**Theorem (Karp 1972):** Determining $\alpha(G)$ is NP-complete.

And the problem remains NP-complete, even if:

1. $G$ is triangle-free

   or

2. $G$ is cubic planar

   or

3. $G$ is $K_{1,4}$-free.

OK! I'm happy to *approximate* $\alpha(G)$

(in polynomial time)!!!

HAH!!!!

You CANNOT approximate closer than

$$1.36\alpha(G)$$

(in polynomial time)

Unless

$$P = NP$$

So when is finding $\alpha(G)$ $easy$ ???

It is trivially *easy* (i.e.,*polynomial*) to find $\alpha(G)$ if

every maximAL independent set is maximUM.

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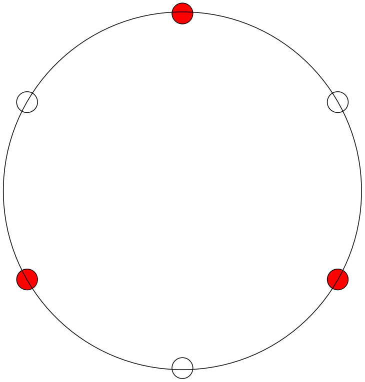
Just start with any vertex and build your independent set in a *greedy* manner!

Graphs with this property are called well-covered.

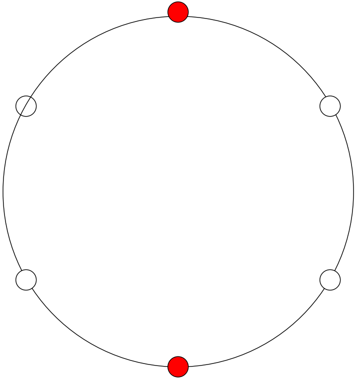
Examples: $C_3, C_4, C_5, C_7$

But NOT $C_6$!

$C_6$



$C_6$



Great!!!!!

.....but.....

When is a graph well-covered?

Can these graphs be recognized in polynomial time???

Well, given a non-well-covered graph $G$, I hand you two maximal independent sets of differing cardinalities. You can check their maximality in polynomial time.

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So recognizing a non-well-covered graph is in co-NP.

Actually, the problem is known to be co-NP-complete!

(Chvátal-Slater (1993); Sankaranarayana-Stewart (1992))

And it remains co-NP-complete, even for

circulant graphs

.

(Brown and Hoshino, 2011)

But the complexity of the recognition problem for graphs that *are* well-covered remains

UNKNOWN!!!

Finbow, Hartnell and Nowakowski (1993) characterized well-covered graphs having

girth at least 5

and their characterization leads to a

polynomial recognition algorithm

So it remains to focus on

girth 3 and 4

PROBLEM (2011):

Characterize well-covered planar quadrangulations

**Lemma:** A planar quadrangulation

(a) contains no triangles

and

(b) is bipartite.

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Part (b) follows from part (a) and induction.

Ravindra's Theorem: A bipartite well-covered graph $G$ contains a perfect matching and for every perfect matching $M$ in $G$ and for every edge $e$ in $M$, $G[N(x) \cup N(y)]$ is a complete bipartite graph.

So in particular, a bipartite well-covered graph must be balanced.

Let us denote by WCQ, the set of all well-covered quadrangulations of the plane.

Theorem: Suppose $G \in WCQ$, $M$ is a perfect matching in $G$ and $e = xy$ is an edge in $M$. Then either $G = C_4$ or else exactly one of $x$ and $y$ has degree 2 in $G$.

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(Hence, if $G \neq C_4$, half the vertices of $G$ have degree 2 and the rest have degree at least 3.)

Now define a second set of quadrangulations of the plane, denoted by $WCQ'$, as follows:

**Def.:** A quadrangulation $Q'$ belongs to $WCQ'$ if there is a set of vertex-disjoint 4-cycles, $C_1, C_2, \ldots, C_k$ in the plane

(we call these *basic* 4-cycles)

such that $V(Q') = V(C_1) \cup \cdots \cup V(C_k)$ and each pair of basic 4-cycles are joined according to the following recipe:

Either the pair are joined by no edges

or

they are joined precisely as shown in Figure 1 below:

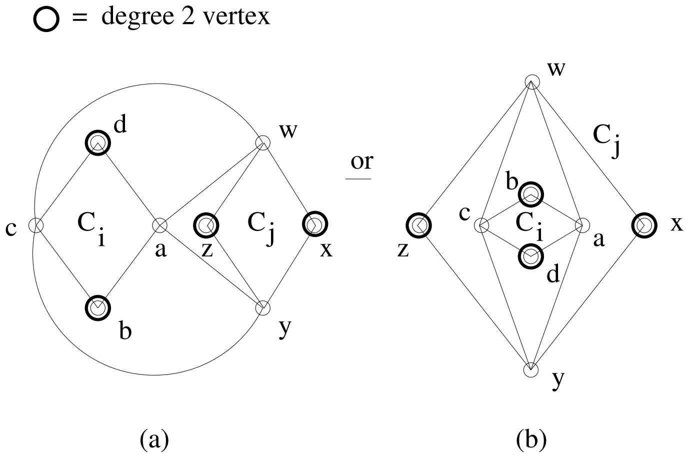
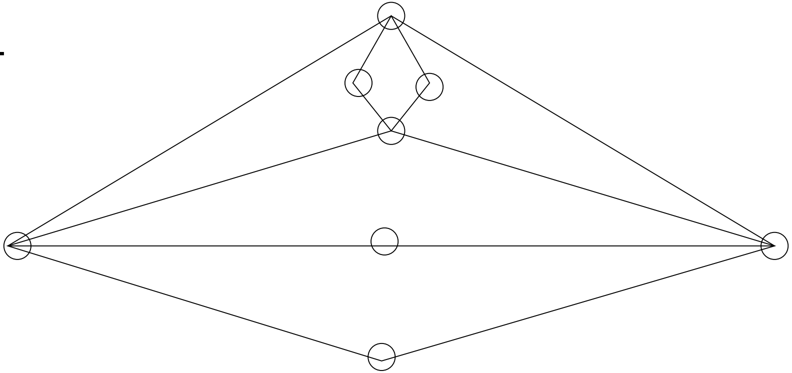



Figure 1

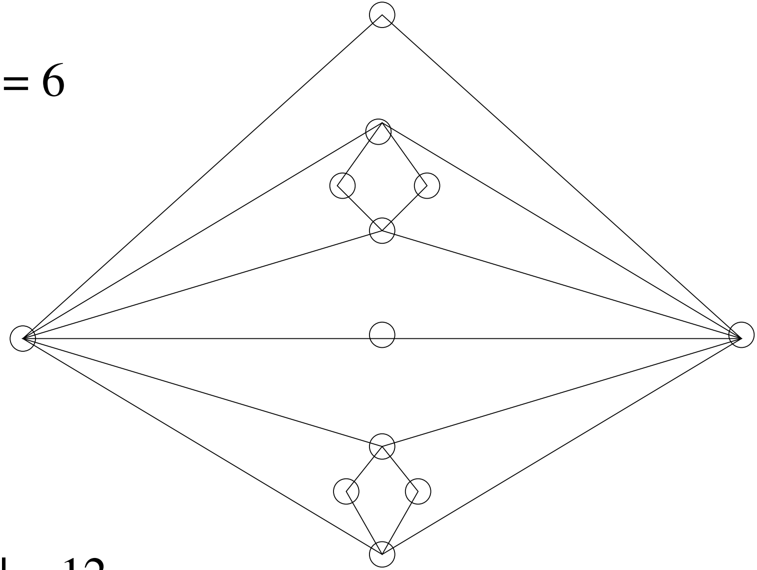
Here are some examples of graphs belonging to $WCQ'$:

$\alpha(G) = 4$



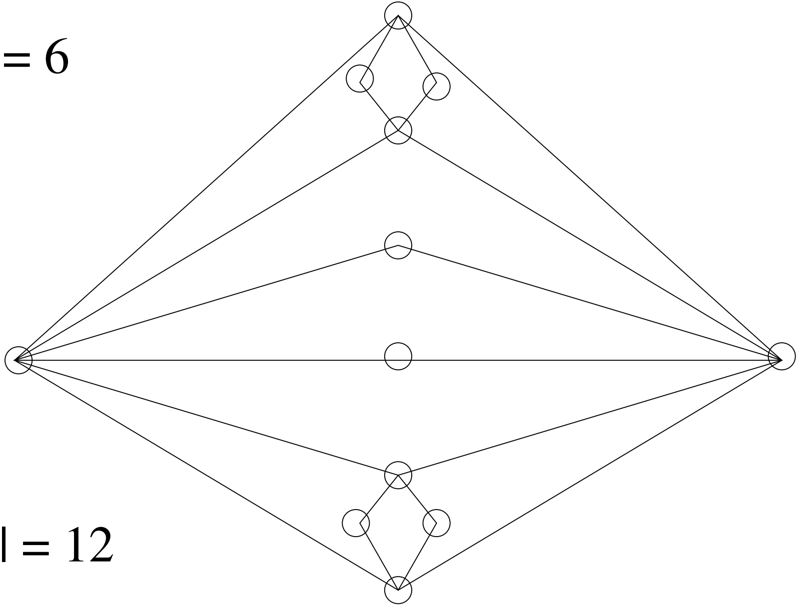
$|V(G)| = 8$

$\alpha(G) = 6$



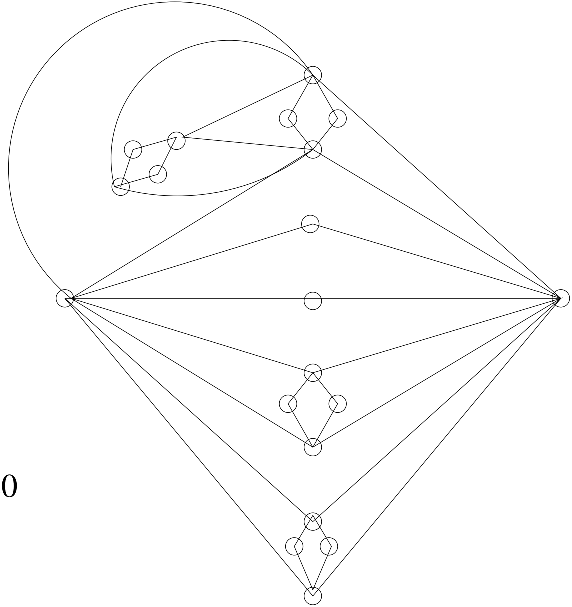
$|V(G)| = 12$

$\alpha(G) = 6$



$|V(G)| = 12$

$\alpha(G) = 10$



$|V(G)| = 20$

Main Theorem: $WCQ = WCQ'$.

Proof: $WCQ \subseteq WCQ'$: Argument uses Ravindra's Theorem repeatedly.

$WCQ' \subseteq WCQ$: If $G = C_4$, this is clear.

If $G \neq C_4$, we argue that any maximum independent set $I$ in $G$ must contain *precisely two* vertices from each basic 4-cycle.

Recognition of graphs in $WCQ$ is clearly *polynomial*.

1. Find a perfect matching $M$.

   (If none exists, $G \notin WCQ$.)

2. By Ravindra's theorem, if $G \neq C_4$, each edge of $M$ must have a vertex of degree 2 in $G$.

   Use $M$ and Ravindra's theorem via the method used in the Main Theorem to build a set of basic 4-cycles.

Note that, if $G \neq C_4$, each basic 4-cycle contains *exactly two* vertices of degree 2. If the process fails, $G$ is not well-covered.

3. Now test every pair of basic 4-cycles to see that either they are joined by no edge or they are joined precisely as in Figure 1 above.

4. If each pair are so joined, $G$ is in $WCQ$.

If there is a pair that are not so joined, $G$ is *not* in $WCQ$.

— THE END —

PROBLEM (1988):

Characterize well-covered planar *triangulations*

This has proved *much harder* than quadrangulations!!!

# A ROADMAP:

1. 5-connected:

There are none!

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(Finbow, Hartnell, Nowakowski +MDP 2004)

2. 4-connected:

There are precisely 4 !

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This was done in *two steps:*

(a) If a 4-connected well-covered triangulation contains two adjacent vertices of degree 4, then there are precisely four such graphs.

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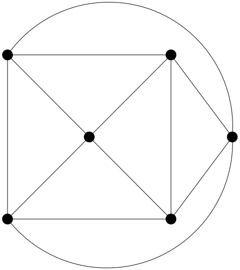
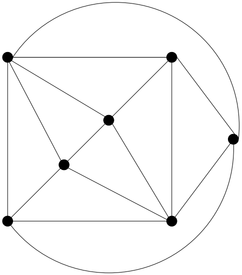
(FHNP 2009)

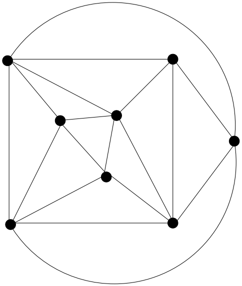
$R_6$



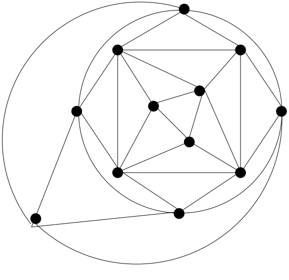
$R_7$



$R_8$



$R_{12}$



(b) Every 4-connected well-covered triangulation must contain two adjacent vertices of degree 4.

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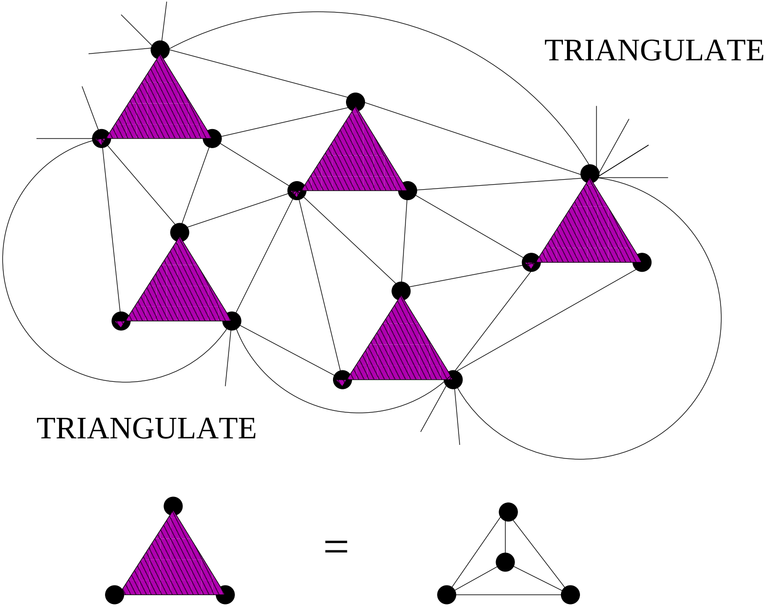
(FHNP 2010)

3. What about 3-connected triangulations??????

Here is an infinite family:


TRIANGULATE
TRIANGULATE
=

The family of such graphs is called the $K_4$-family and is denoted by $\mathcal{K}$.

BUT.....these are NOT ALL!!!

*Flash!!!*

The family is now characterized and is polynomially recognizable (FHNP 2012).

The paper is some 40 pages long (!), so we will give just an outline:

Lemma: If $G$ is well-covered and $v$ is a vertex in $G$, then $G - N[v]$ is well-covered.

Applying this lemma repeatedly, it is easy to see that

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Lemma: If $G$ is well-covered and $I = \{v_1, \ldots, v_k\}$ is an independent set in $G$, then $G - N[I] = G - (\cup_{i=1}^{k} N[v_i])$ is also well-covered.

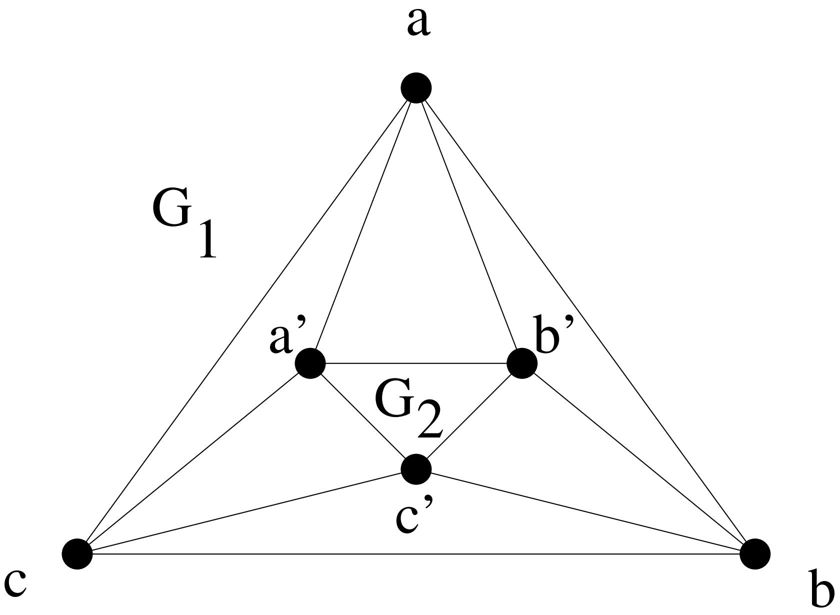
We often use the preceding lemma to show that a certain graph is *not* well-covered, by strategically finding an independent $I$ in $G$ such that $G - N[I]$ is *not* well-covered and therefore the parent graph is *not* well-covered.

BUT it can be very difficult to find just the right independent set $I$ here!

Next we need a new concept called O-join.

Suppose that $G_1$ and $G_2$ are both 3-connected planar triangulations and that $G_1$ contains a triangular face $abca$ and $G_2$, a triangular face $a'b'c'a'$. Embed $G_1$ so that $abca$ is an *interior* face and embed $G_2'$ so that $a'b'c'a'$ bounds the *infinite* face.

Let$G_1 \bigcirc G_2$ denote the graph obtained by embedding $G_2$ into the interior of face *abca* of $G_1$ and adding the six edges shown in the following figure.


a
G
1
a’
b’
G
2
c’
c
b

Then $G_1 \bigcirc G_2$ is called an O-join of $G_1$ and $G_2$ at the faces $abca$ and $a'b'c'a'$.

(The "O" in "O-join" stands for "octahedral".)

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(Note that given two triangles labeled as above, there are *six* possible O-joins at these triangles.)

Theorem: If $G_1$ and $G_2$ are each 3-connected planar well-covered triangulations, then any O-join $G_1 \bigcirc G_2$ is also a 3-connected planar well-covered triangulation.

The converse of this theorem is

MUCH MORE DIFFICULT!!

In fact, most of this long paper is devoted to showing that:

if $G$ is a 3-connected planar well-covered triangulation and $G$ is not one of ten exceptional graphs, then $G$ must be constructed from smaller members of the family via a succession of O-joins.

**Def.:** Let $G$ be a well-covered triangulation and $abca$, a face of $G$. Then $abca$ is called a YES-face if $G-a-b, G-a-c$ and $G-b-c$ is also well-covered.

A triangular face which is not a YES-face is called a NO-face.

Lemma: Suppose $G_1$ and $G_2$ are planar triangulations O-joined at triangles $T_1$ and $T_2$, respectively, to yield $G = G_1 \bigcirc G_2$.

Then $G$ is well-covered if and only if

(1) $G_1$ and $G_2$ are both well-covered, and

(2) $T_i$ bounds a YES-face in $G_i$, for $i = 1$ and 2.

Also, if $G$ is well-covered, then $\alpha(G) = \alpha(G_1) + \alpha(G_2)$.

## SOME EXAMPLES:

(1) Both faces of $K_3$ are YES-faces and all faces of $K_4$ are YES-faces.

(2) If a triangle $K_3$ with vertices $x, y$ and $z$ is O-joined to a w.-c. graph $G$ via its YES-face $abca$, to obtain a graph $H$, then the six faces generated in taking the O-join, together with the original $K_3$ form a set of seven NO-faces.

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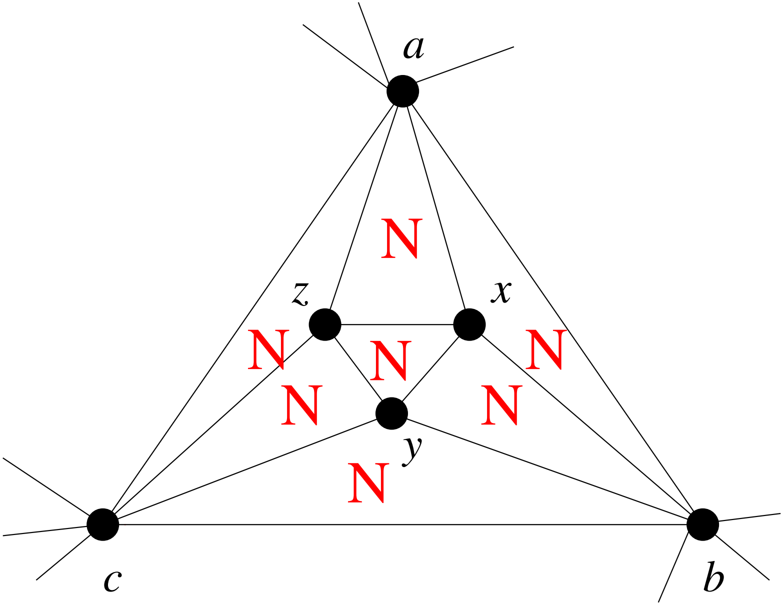
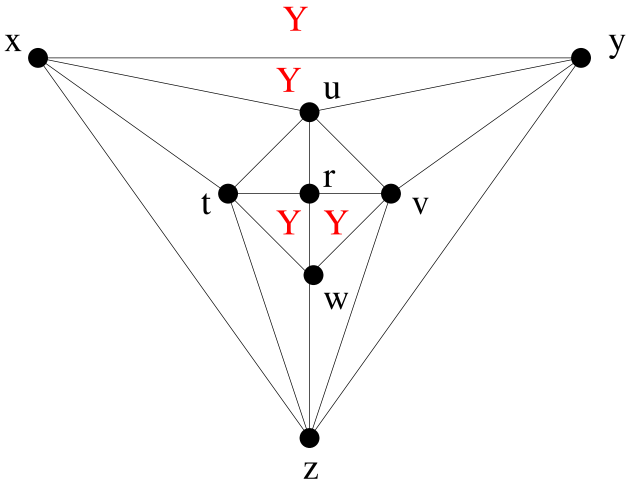
(See the next figure:)



a
z
x
y
c
b
N
N
N
N
N
N
N

(3) In $R_6, R_7$ and $R_{12}$, each triangular face is a NO-face.

(4) In $R_8$, the four faces labeled “Y” in the following figure are the only YES-faces.




$R_8$

(5) Informally, a YES-face is one at which one can O-join another w-c. triangulation and, in the process, obtain a new w-c. triangulation!

Next we consider:

# Well-covered triangulations having

# NO

# O-joins

**Def.:** A vertex in a graph $G$ is white if $\deg_G(v) = 3$ or $v$ is adjacent to a vertex with degree 3.

(NOTE: In a well-covered triangulation, no two different $K_4$s can share a vertex!)

**Def.:** Let us call a non-white vertex blue.

At this point, we show that:

(1) If a w-c. triangulation $G$ contains a white vertex, but *no* O-joins, then it belongs to $\mathcal{K}$; that is, *all* the vertices of $G$ are white.

(2) If a w-c. triangulation $G$ contains *no* white vertex and *no* O-joins, then

$$G \in \{K_3, R_7, R_8, R_{12}\}.$$

If $G$ is a w-c. triangulation containing at least one white vertex, at least one blue vertex and has no O-joins, then we call $G$ *bad*.

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The bulk of the paper is then devoted to showing:

There is NO BAD triangulation.

This is done by considering a bad graph of minimum size.

To summarize:

**Def.:** The extended $K_4$-family, denoted $\mathcal{K}^+$, is:

(a) the collection of all graphs that can be obtained from a plane triangulation $G$, a member of the $K_4$-family $\mathcal{K}$ having at least five vertices, by choosing two disjoint sets $R$ and $S$ (possibly empty) of YES-faces in $G$ and O-joining a triangle to each face in $R$ and O-joining a copy of $R_8$ to each face of $S$ via an appropriate YES-face of $R_8$, together with

(b) $K_4, K_4 \bigcirc K_3$ and $K_4 \bigcirc R_8$.

We can now state our characterization as follows:

**Characterization Theorem:** Let $G$ be a planar triangulation. Then $G$ is well-covered if and only if $G$ belongs to the extended $\mathcal{K}_4$-family or else $G$ is one of the following graphs:

$$K_3, R_6, R_7, R_8, R_{12}, R_8 \bigcirc K_3 \text{ or } R_8 \bigcirc R_8.$$

A well-covered planar triangulation is either one of ten special graphs

or it must have come from two smaller well-covered triangulations via an O-join.

One then looks for new O-joins in the two smaller component graphs and continue until the component graphs are O-join-free.

Since there can be at most a polynomial number of O-joins in a planar triangulation, we have a polynomial algorithm for recognizing planar well-covered triangulations.