Disjoint chorded cycles of the same length

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- Egawa (1996): Confirmed Thomassen's conjecture for $k \geq 3$ and $n_k = 17k + o(k)$.
- Verstraëte (2003): Gave a proof for all k for Thomassen's conjecture but with a larger n_k.

History Cont. - Chorded Cycles

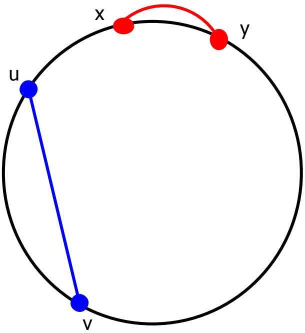

Figure: A cycle with two chords xy and uv

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- Recall: Verstraëte, G with $|G| \geq n_k$ and $\delta(G) \geq 2k$ contains k vertex-disjoint cycles of the same length.
- Question: What about k vertex-disjoint chorded cycles of the same length?

Main Result 1

Theorem

Let k be a natural number. Then there exists a positive integer n_k such that if G is a graph of order at least n_k and minimum degree at least $3k+8$, then G contains k vertex-disjoint chorded cycles of the same length.

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Qusetion

Can we change the minimum degree condition to $3k$? (Best possible, $K_{3k-1,n-3k+1}$.)

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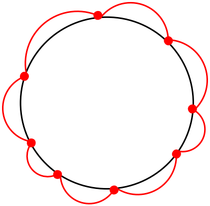

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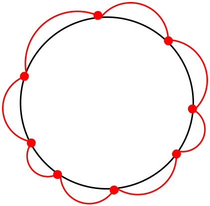

Figure: $\delta = \overline{d} = 4$, no “short” chorded cycles when $n > 300\log_2 n$

??? For simple graph with minimum degree at least 3, can we get a similar result? What about average degrees?

Main Result 2 – A Chorded Version of A Result of Bollobás and Thomason

Theorem (Bollobás and Thomason)

Let G be a graph of order n and size at least $n+c\,(c \geq 1)$. Then $g(G) \leq 2(\lfloor n/c \rfloor + 1)\lfloor \log_2 2c \rfloor$.

Proof Ideas for Main Result 1 – A Lemma

Lemma (Verstraëte, 2003)

Let $G(A,B)$ be a bipartite graph with $|A| > a|B|^b$. Suppose that $d(v) \geq b$ for all $v \in A$. Then for some $r \geq 1$, there exists $A_1, A_2, \cdots, A_r, W \subset A$ and sets $B_1, B_2, \cdots, B_r \subset B$ such that for each $i = 1, 2, \cdots, r$, $G(A_i, B_i) = G[A_i \cup B_i]$ is a complete bipartite graph with $|A_i| \geq a$ and $|B_i| = b$, the B_i are distinct, the A_i are disjoint, $A = A_1 \cup A_2 \cup \cdots \cup A_r \cup W$ and $|W| < a|B|^b$.

Observation: If $G(A,B)$ is a bipartite graph with $|A| > 3|B|^3 + 3(k-1)$ such that $d(v) \geq 3k$ for all $v \in A$. Then, by applying the above lemma k times, we can find k vertex-disjoint copies of $K_{3,3}$, that is, k vertex-disjoint chorded 6-cycles.

Proof ideas for Main Result 1

Recall: G with $|G| \geq n_k$ and $\delta(G) \geq 3k+8 \Rightarrow k$ vertex-disjoint chorded cycles of the same length.

- Set $n_k = \min\{n \in \mathbb{N} : n > 28k(k-1)^{3k}\left(\frac{301^2}{2}\right)^{3k}(\log_2 n)^{6k}\}$.

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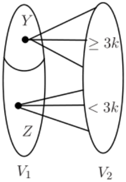
- May assume $\mathcal{C}$ has no k vertex-disjoint chorded cycles of the same length.
- $|V_2| \leq (k-1)\frac{301^2}{2}(\log_2 n)^2$.

Proof ideas for Main Result 1 Cont.

Let $V_1 = V - V_2$,
$Y = \{v \in V_1 \mid d_{V_2}(v) \geq 3k\}$ and $Z = \{v \in V_1 \mid d_{V_2}(v) < 3k\}$



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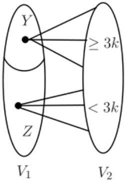

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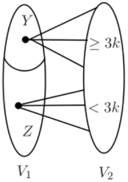

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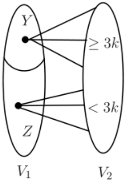

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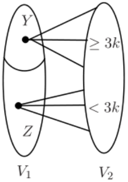

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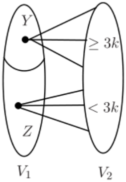

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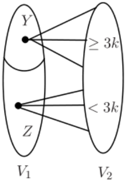

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- G' contains a subgraph H of minimum degree at least 5.
- By Main Result 2 $\Rightarrow$ a chorded cycle of length at most $300 \log_2 n$ in H, and which is disjoint with cycles in $\mathcal{C} \Rightarrow$ CONTRADICTION.

Thank You