3-flow in 8-edge-connected signed graphs

Yezhou Wu
The University of Hong Kong
Joint work with D.Ye, W.Zang and C.-Q.Zhang

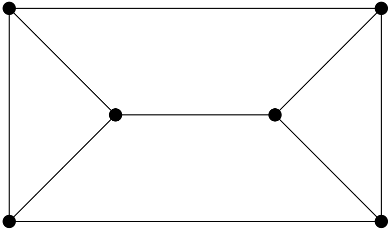
05 25 2013

k-face-color

- Given a planar graph G, a k-face-coloring is a mapping $f: \{\text{faces}\} \to \{0, 1, \cdots, k-1\}$ such that no two adjacent faces have the same color.

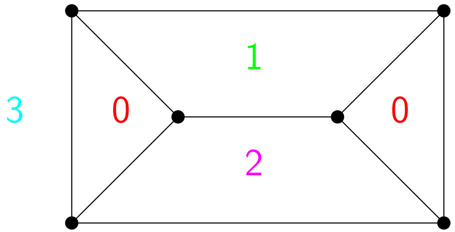
k-face-color

- Given a planar graph G, a k-face-coloring is a mapping $f:\ \{\text{faces}\} \to \{0, 1, \cdots, k-1\}$ such that no two adjacent faces have the same color.



k-face-color

- Given a planar graph G, a k-face-coloring is a mapping $f: \{\text{faces}\} \to \{0, 1, \cdots, k-1\}$ such that no two adjacent faces have the same color.



Tutte's integer flow

Integer flow was originally introduced by Tutte(1949) as a generalization of map coloring problems.

Tutte's integer flow

Integer flow was originally introduced by Tutte(1949) as a generalization of map coloring problems.

A mapping $f : E(G) \mapsto \mathbb{Z}$ is a nowhere-zero k-flow(k-NZF) if under orientation D,

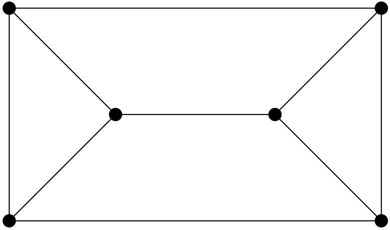
- (1) $\sum_{e\in E^+(v)} f(e) = \sum_{e\in E^-(v)} f(e)$;
- (2) $0 < |f(e)| < k$ for every edge $e \in E(G)$.

Tutte's integer flow

Integer flow was originally introduced by Tutte(1949) as a generalization of map coloring problems.

A mapping $f : E(G) \mapsto \mathbb{Z}$ is a nowhere-zero k-flow(k-NZF) if under orientation D,

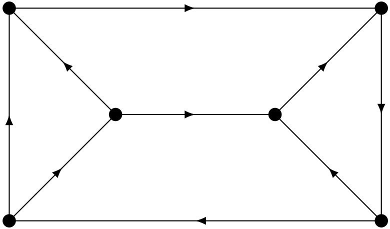
- (1) $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$;
- (2) $0 < |f(e)| < k$ for every edge $e \in E(G)$.



Integer flow was originally introduced by Tutte(1949) as a generalization of map coloring problems.

A mapping $f : E(G) \mapsto \mathbb{Z}$ is a nowhere-zero k-flow(k-NZF) if under orientation D,

- (1) $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$;
- (2) $0 < |f(e)| < k$ for every edge $e \in E(G)$.

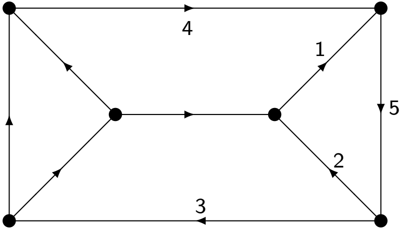


Tutte's integer flow

Integer flow was originally introduced by Tutte(1949) as a generalization of map coloring problems.

A mapping $f : E(G) \mapsto \mathbb{Z}$ is a nowhere-zero k-flow(k-NZF) if under orientation D,

- (1) $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$;
- (2) $0 < |f(e)| < k$ for every edge $e \in E(G)$.

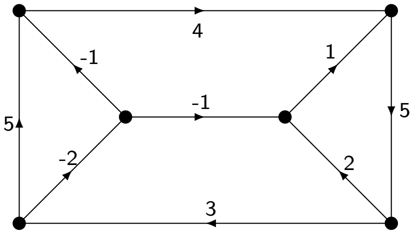


Tutte's integer flow

Integer flow was originally introduced by Tutte(1949) as a generalization of map coloring problems.

A mapping $f : E(G) \mapsto \mathbb{Z}$ is a nowhere-zero k-flow(k-NZF) if under orientation D,

- (1) $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$;
- (2) $0 < |f(e)| < k$ for every edge $e \in E(G)$.



Tutte's integer flow

Theorem Tutte's Theorem Let G be a 2-edge-connected planar graph.

$$G \text{ is } k\text{-face-colorable} \quad \Leftrightarrow \quad G \text{ has a nowhere-zero } k\text{-flow.}$$

Tutte's integer flow

Theorem Tutte's Theorem Let G be a 2-edge-connected planar graph.

G is k-face-colorable $\Leftrightarrow$ G has a nowhere-zero k-flow.

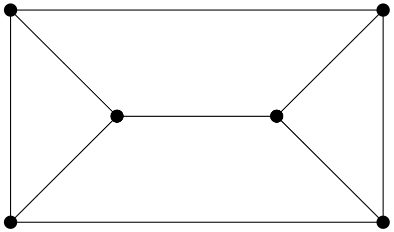
A planar graph G is 4-face-colorable $\Leftrightarrow$ G has a 4-NZF.

Tutte's integer flow

Theorem Tutte's Theorem Let G be a 2-edge-connected planar graph.

G is k-face-colorable $\Leftrightarrow$ G has a nowhere-zero k-flow.

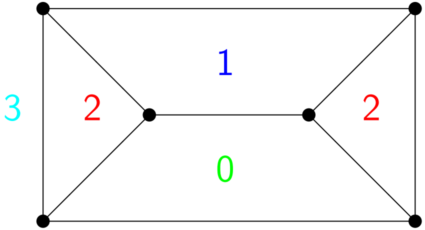
A planar graph G is 4-face-colorable $\Leftrightarrow$ G has a 4-NZF.



Theorem Tutte's Theorem Let G be a 2-edge-connected planar graph.

G is k-face-colorable $\Leftrightarrow$ G has a nowhere-zero k-flow.

A planar graph G is 4-face-colorable $\Leftrightarrow$ G has a 4-NZF.

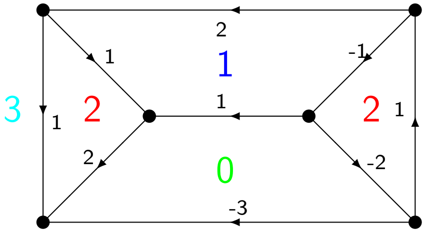


Tutte's integer flow

Theorem Tutte's Theorem Let G be a 2-edge-connected planar graph.

G is k-face-colorable $\Leftrightarrow$ G has a nowhere-zero k-flow.

A planar graph G is 4-face-colorable $\Leftrightarrow$ G has a 4-NZF.



Tutte's flow conjectures

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF;

Tutte's flow conjectures

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF;
- 4-Flow Conjecture Every bridgeless graph containing no subdivision of the Petersen graph has a 4-NZF;

Tutte's flow conjectures

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF;
- 4-Flow Conjecture Every bridgeless graph containing no subdivision of the Petersen graph has a 4-NZF;
- 5-Flow Conjecture Every bridgeless graph has a 5-NZF.

Tutte's flow conjectures

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF;
- 4-Flow Conjecture Every bridgeless graph containing no subdivision of the Petersen graph has a 4-NZF;
- 5-Flow Conjecture Every bridgeless graph has a 5-NZF.

They are three major problems in graph theory.

All remain open!

3-Flow conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF;

3-Flow conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF;

3-Flow Conjecture is true for planar graphs.

3-Color Theorem (Grötzsch 1958) Every 4-edge-connected planar graph is face 3-colorable ($\Longleftrightarrow$ has a 3-NZF).

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.

Let G be a graph of order n with n_o odd vertices.

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.

Let G be a graph of order n with n_o odd vertices.

- Theorem(Lai and Zhang, DM1992)
  Every $4\lceil \log_2 n_o \rceil$-edge-connected has a 3-NZF.
- Theorem(Alon, Linial and Meshulam, JCTA1991)
  Every $2\lceil \log_2 n \rceil$-edge-connected has a 3-NZF.
  See also (Alon and Pralat, CPC 2010).
  CPC: Combinatorics, Probability and Computing.

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.

- Theorem(Thomassen, JCTB2012)
  Every 8-edge-connected graph has a 3-NZF.

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.

- Theorem(Thomassen, JCTB2012)
  Every 8-edge-connected graph has a 3-NZF.
  $\Downarrow$
  6 (Lov$\acute{a}$sz, Thomassen, Wu and Zhang, JCTB submitted)

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.

- Theorem(Thomassen, JCTB2012)
  Every 8-edge-connected graph has a 3-NZF.

  $\Downarrow$

  6 (Lov$\acute{a}$sz, Thomassen, Wu and Zhang, JCTB submitted)

Just **1-step** from the 3-Flow Conjecture.

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.

- Theorem(Thomassen, JCTB2012)
  Every 8-edge-connected graph has a 3-NZF.
  $\Downarrow$
  6 (Lov$\acute{a}$sz, Thomassen, Wu and Zhang, JCTB submitted)

Just **1-step** from the 3-Flow Conjecture.

Theorem(Kochol, JCTB2001)
Smallest counterexample is 5-edge-connected.

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.
- Theorem(Thomassen, JCTB2012)
  Every 8-edge-connected graph has a 3-NZF.

Weak 3-Flow Conjecture

- 3-Flow Conjecture Every 4-edge-connected graph has a 3-NZF.
- Weak 3-Flow Conjecture(Jaeger, JCTB1979) There is an integer h such that every h-edge-connected graph has a 3-NZF.

- Theorem(Thomassen, JCTB2012)
  Every 8-edge-connected graph has a 3-NZF.

$$\Downarrow$$

signed graph (Wu, Ye, Zang and Zhang)

What is a signed graph?

A *signed graph* is a pair (G, σ) such that

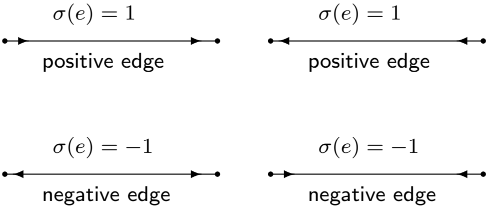
G: a graph;
$\sigma : E(G) \to \{1, -1\}$.

What is a signed graph?

A *signed graph* is a pair (G, σ) such that

G: a graph;

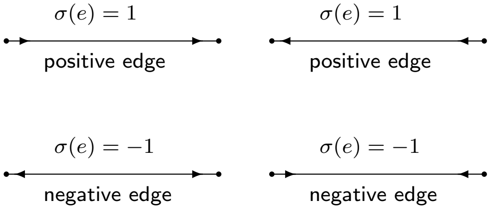
$\sigma : E(G) \to \{1, -1\}$.



What is a signed graph?

A *signed graph* is a pair (G, σ) such that

G: a graph;
$\sigma : E(G) \to \{1, -1\}$.



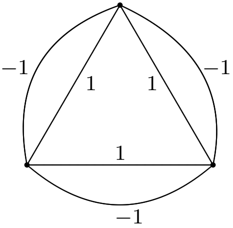
A graph $G \iff$ a signed graph (G, σ) with $\sigma = 1$.

What is a signed graph?

A *signed graph* is a pair (G, σ) such that

G: a graph;

$\sigma : E(G) \to \{1, -1\}$.

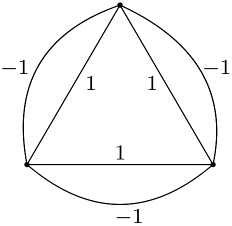


signed graph (G, σ)

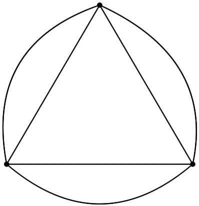
What is a signed graph?

A *signed graph* is a pair (G, σ) such that

G: a graph;
$\sigma : E(G) \to \{1, -1\}$.



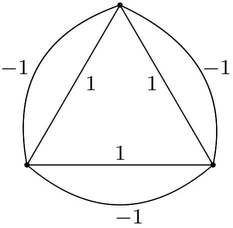
signed graph (G, σ)



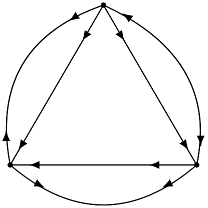
What is a signed graph?

A *signed graph* is a pair (G, σ) such that

G: a graph;
$\sigma : E(G) \to \{1, -1\}$.



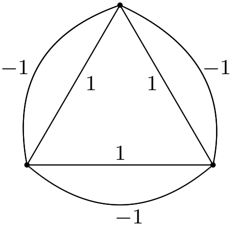
signed graph (G, σ)



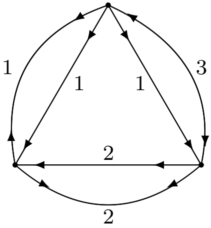
What is a signed graph?

A *signed graph* is a pair (G, σ) such that

G: a graph;
$\sigma : E(G) \to \{1, -1\}$.



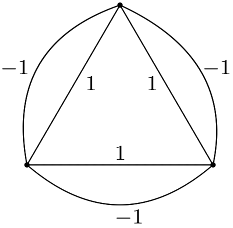
signed graph (G, σ)



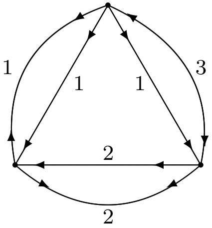
What is a signed graph?

A *signed graph* is a pair (G, σ) such that

G: a graph;
$\sigma : E(G) \to \{1, -1\}$.



signed graph (G, σ)



(G, σ) has a 4-NZF

A necessary condition

Theorem(Tutte) Every bridgeless graph has a k-NZF.

A necessary condition

Theorem(Tutte) Every bridgeless graph has a k-NZF.

$\Downarrow$

8 (Jaeger)

A necessary condition

Theorem(Tutte) Every bridgeless graph has a k-NZF.

$\Downarrow$ 8 (Jaeger)

$\Downarrow$ 6 (Seymour)

A necessary condition

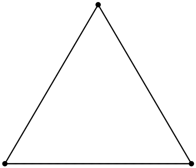
Theorem(Tutte) Every bridgeless graph has a k-NZF.

$\Downarrow$

8 (Jaeger)

$\Downarrow$

6 (Seymour)



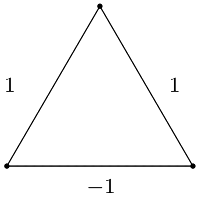
G has has 2-NZF

A necessary condition

Theorem(Tutte) Every bridgeless graph has a k-NZF.

$\Downarrow$ 8 (Jaeger)

$\Downarrow$ 6 (Seymour)



A necessary condition

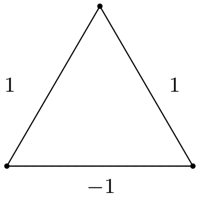
Theorem(Tutte) Every bridgeless graph has a k-NZF.

$\Downarrow$

8 (Jaeger)

$\Downarrow$

6 (Seymour)

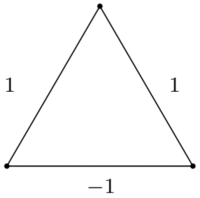


(G, σ) doesn't has any NZF

A necessary condition

Theorem(Tutte) Every bridgeless graph has a k-NZF.

$\Downarrow$
8 (Jaeger)
$\Downarrow$
6 (Seymour)



(G, σ) doesn't has any NZF

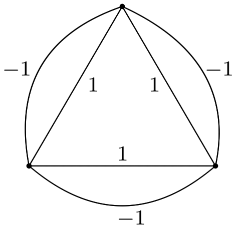
Conjecture(Bouchet, JCTB 1983)
If a signed graph has a NZF, then it has a 6-NZF.

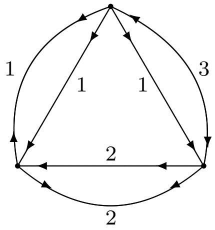
3-NZF of Signed Graphs

Theorem(Wu, Ye, Zang and Zhang, preprint) Every 8-edge-connected signed graph with a NZF has a 3-NZF.

3-NZF of Signed Graphs

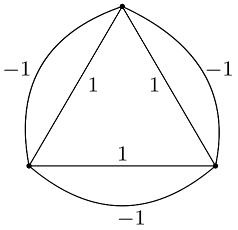
Theorem(Wu, Ye, Zang and Zhang, preprint) Every 8-edge-connected signed graph with a NZF has a 3-NZF.

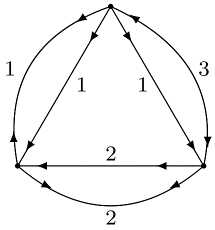




3-NZF of Signed Graphs

Theorem(Wu, Ye, Zang and Zhang, preprint) Every 8-edge-connected signed graph with a NZF has a 3-NZF.





(G, σ) is 4-edge-connected and has a 4-NZF but no 3-NZF.
So 3-Flow Conjecture can not be generalized to signed graphs.

3-NZF of Signed Graphs

Theorem(Wu, Ye, Zang and Zhang, preprint) Every 8-edge-connected signed graph with a NZF has a 3-NZF.

3-NZF of Signed Graphs

Theorem(Wu, Ye, Zang and Zhang, preprint) Every 8-edge-connected signed graph with a NZF has a 3-NZF.

Conjecture Every 5-edge-connected signed graph with a NZF has a 3-NZF.

Theorem(Wu, Ye, Zang and Zhang, preprint) Every 8-edge-connected signed graph with a NZF has a 3-NZF.

Conjecture Every 5-edge-connected signed graph with a NZF has a 3-NZF.

$\Downarrow$

Every 5-edge-connected graph has a 3-NZF.

3-NZF of Signed Graphs

Theorem(Wu, Ye, Zang and Zhang, preprint) Every 8-edge-connected signed graph with a NZF has a 3-NZF.

Conjecture Every 5-edge-connected signed graph with a NZF has a 3-NZF.

$\Downarrow$

Every 5-edge-connected graph has a 3-NZF.

$\Downarrow$

3-Flow Conjecture

Modulo $(2p+1)$-Orientations

- Definition An orientation D of G is a **modulo** $(2p+1)$-**orientation** if at $\forall v$,
$$d_D^+(x) - d_D^-(x) \equiv 0 \pmod{(2p+1)}.$$

Modulo $(2p + 1)$-Orientations

- Definition An orientation D of G is a **modulo** $(2p + 1)$-**orientation** if at $\forall v$,
$$d^+_D(x) - d^-_D(x) \equiv 0 \pmod{(2p+1)}.$$

Theorem A graph has a $(2 + 1/p)$-NZF if and only if it has a modulo $(2p + 1)$-orientation.

Modulo $(2p+1)$-Orientations

- Definition An orientation D of G is a **modulo** $(2p+1)$-**orientation** if at $\forall v$,

$$d_D^+(x) - d_D^-(x) \equiv 0 \pmod{(2p+1)}.$$

Theorem A graph has a $(2+1/p)$-NZF if and only if it has a modulo $(2p+1)$-orientation.

The definition of **modulo** $(2p+1)$-**orientation** can be generalized to signed graphs.

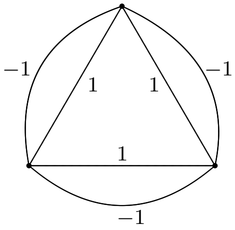
Theorem(Xu and Zhang, DM2005) A siged graph has a 3-NZF if and only if it has a modulo 3-orientation.

Modulo 3-Orientations

Theorem(Xu and Zhang, DM2005) A signed graph has a 3-NZF if and only if it has a modulo 3-orientation.

Modulo 3-Orientations

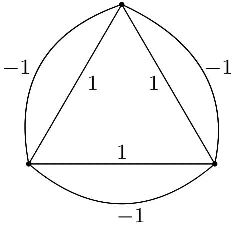
Theorem(Xu and Zhang, DM2005) A signed graph has a 3-NZF if and only if it has a modulo 3-orientation.



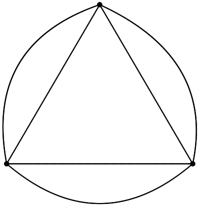
signed graph (G, σ)

Modulo 3-Orientations

Theorem(Xu and Zhang, DM2005) A signed graph has a 3-NZF if and only if it has a modulo 3-orientation.

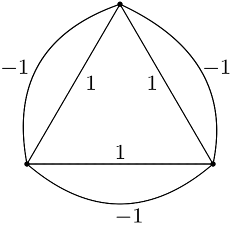


signed graph (G, σ)

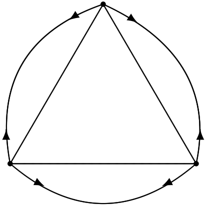


Modulo 3-Orientations

Theorem(Xu and Zhang, DM2005) A signed graph has a 3-NZF if and only if it has a modulo 3-orientation.

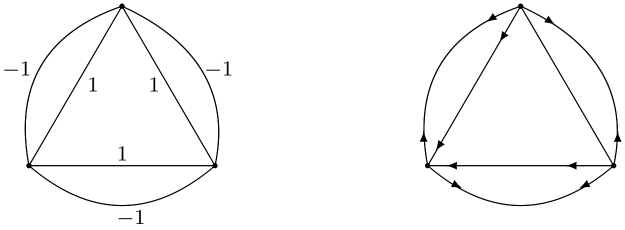


signed graph (G, σ)



Modulo 3-Orientations

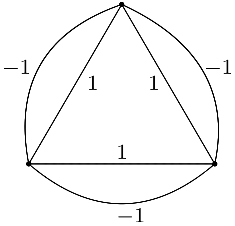
Theorem(Xu and Zhang, DM2005) A signed graph has a 3-NZF if and only if it has a modulo 3-orientation.



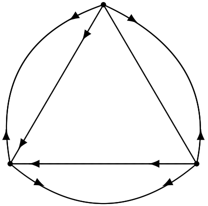
signed graph (G, σ)

Modulo 3-Orientations

Theorem(Xu and Zhang, DM2005) A signed graph has a 3-NZF if and only if it has a modulo 3-orientation.



signed graph (G, σ)



(G, σ) has no 3-NZF

Flows in graphs and signed graphs

(κ, ϕ) : Every κ-edge-connected (signed) graph has a ϕ-NZF.

Flows in graphs and signed graphs

(κ, ϕ) : Every κ-edge-connected (signed) graph has a ϕ-NZF.

Graphs:

(2,6) (Seymour, JCTB 1992) (4,4) (Jaeger, JCTB 1979)

(6,<4) (Galluccio and Goddyn, Combinatorica 2002 $\&$ Lai, Xu and Zhang, Combinatorica 2007)

(8,3) and $(8p^2 + 10p + 8$,2+1/p) (Thomassen, JCTB 2012)

(6p,2+1/p) (Lovász, Thomassen, Wu and Zhang, JCTB submitted)

Flows in graphs and signed graphs

(κ, ϕ) : Every κ-edge-connected (signed) graph has a ϕ-NZF.

Graphs:

(2,6) (Seymour, JCTB 1992) (4,4) (Jaeger, JCTB 1979)
(6,<4) (Galluccio and Goddyn, Combinatorica 2002 & Lai, Xu and Zhang, Combinatorica 2007)
(8,3) and ($8p^2 + 10p + 8$,2+1/p) (Thomassen, JCTB 2012)
(6p,2+1/p) (Lovász, Thomassen, Wu and Zhang, JCTB submitted)

Signed Graphs:

(2,12) (Devos, manuscript 2004)
(4,4) and (6,<4) (Raspaud and Zhu, JCTB 2011)
(11p, 2+1/p) (Zhu, preprint).
(8,3) (Wu, Ye, Zang and Zhang, preprint)

Thank you!