3-Factor-criticality of vertex-transitive graphs

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Vertex-transitive graphs

Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. Only finite graphs with no loops and parallel edges are considered in this talk.

A graph G is said to be vertex-transitive if for any two vertices x and y in G there is an automorphism φ of G such that $y = \varphi(x)$, That is, the automorphism group of G can act transitively on the vertex-set of G

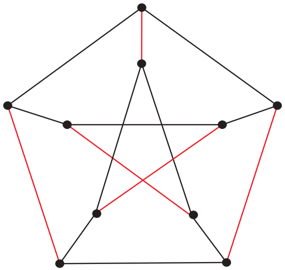
Vertex-transitive graphs include all Cayley graphs, and have been separately studied in group theory, graph theory and networks. In group theory, the automorphism group of vertex-transitive graphs is an algebraic object which is studied. In graph theory, Hamiltonicity and matching extendability of vertex-transitive graphs were studied. In network, the restricted edge-connectivity and cyclic edge-connectivity of vertex-transitive graphs were studied recently.

Perfect matching and Tutte's theorem

A perfect matching of G is a set of independent edges covering all the vertices in G. Petersen's Theorem (1890's): Any bridgeless cubic graph has a perfect matching.

Theorem 1.1 (Tutte, 1947)

A graph G has a perfect matching if and only if $c_0(G - X) \leq |X|$ for any $X \subseteq V(G)$.



Factor-critical graphs and bicritical graphs

The concepts of factor-critical graphs, bicritical graphs and n-extendable graphs were introduced by Gallai in 1963, by Lovász in 1972, and by Plummer in 1980 respectively.

♦ A graph G is called factor-critical if the removal of any vertex of G results in a graph with a perfect matching.

♦ A graph is called bicritical if the removal of any pair of distinct vertices of G results in a graph with a perfect matching.

♦ A graph G is said to be k-extendable if it is connected, has a set of k independent lines and every set of k independent lines in G extends to (i.e. is a subset) a perfect matching of G. $(2k < |V(G)|)$

Factor-criticality and bicriticality of vertex-transitive graphs

Theorem 1.2 (Lovász and Plummer, 1986)

If G is a connected vertex-transitive graph of order n, then
(a) if n is odd, G is factor-critical, while
(b) if n is even, G is either elementary bipartite or bicritical.

Theorem 1.2.(a) can be an immediate consequence of Gallai-Edmonds Structure Theorem. The proof of Theorem 1.2(b) involves the edge-connectivity and super edge-connectivity of vertex-transitive graphs.

L. Lovász, M.D. Plummer, Matching Theory, North-Holland, Amsterdam, 1986.

p-factor-critical graphs

The concepts of factor-critical and bicritical graphs were generalized to the concept of p-factor-critical graphs by Favaron in 1996 and by Yu in 1993, independently. A graph G is said to be p-factor-critical, where p is an integer of the same parity as n, if the removal of any set of p vertices results in a graph with a perfect matching.

Theorem 1.3 (Yu, 1993; Favaron,1996)

A graph G is p-factor-critical if and only if $c_0(G-X) \leq |X| - p$ for every $X \subseteq V(G)$ with $|X| \geq p$.

Theorem 1.4 (Favaron, 1996,2000)

(a)Every p-factor-critical graph of order n $(p < n)$ is $(p-2)$-factor-critical,

(b)For even p, every non-bipartite p-extendable graph is p-factor-critical.

p-factor-critical graphs

For a connected graph G, a vertex-cut of G is a set of vertices whose removal disconnects G. The (vertex-)connectivity of G which is not complete graph is the minimum cardinality over all the vertex-cuts of G, denoted by $\kappa(G)$.

An edge subset F is called an edge-cut of G if $G - F$ is disconnected. The edge-connectivity of G is the minimum cardinality over all edge-cuts of G, denoted by $\lambda(G)$.

Lemma 1.5 (Favarvon, 1996)

If a graph G is p-factor-critical with $1 \leq p < |V(G)|$, then $\kappa(G) \geq p$ and $\lambda(G) \geq p + 1$.

The main result

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Question: What about p-factor-criticality for vertex-transitive graphs for some integer p with $p \geq 3$?

In case of $p = 3$ we answer the question and obtain the following result.

Theorem 1.6

A connected vertex-transitive graph of odd order at least 5 is 3-factor-critical if and only if it is not a cycle.

To prove this, we apply the vertex-connectivity, edge-connectivity and several conditional edge-connectivities of vertex-transitive graphs, which will be introduced in detail in Section 2.

Connectivity of vertex-transitive graphs

Watkins studied the connection between connectivity and vertex-degree for vertex-transitive graphs.

Lemma 2.1 (Watkins,1970)

Let G be a connected k-regular vertex-transitive graph. Then $\kappa(G) > \frac{2}{3}k$.

Lemma 2.2 (Watkins,1970)

If G is vertex-transitive with degree $k = 4$ or 6, then $\kappa(G) = k$.

M.E. Watkins, Connectivity of transitive graphs, J. Combin. Theory 8 (1970) 23-29.

Edge-connectivity of vertex-transitive graphs

Let $\delta(G)$=min$\{d(v) : v \in V(G)\}$. It is well-known that $\lambda(G) \leq \delta(G)$. If $\lambda(G) = \delta(G)$, then G is said to be maximally edge-connected.

Lemma 2.3 (Mader,1971)

All connected vertex-transitive graphs are maximally edge-connected.

W. Mader, Minimale n-fach kantenzusammenhängenden Graphen, Math. Ann. 191 (1971) 21-28.

Super edge-connectivity of vertex-transitive graphs

A connected graph G is said to be super edge-connected, in short, super-λ, if each of its minimum edge-cut is $\nabla(v)$ for some $v \in V(G)$, the set of edges incident to v.

An imprimitive block of G is a proper non-empty subset X of $V(G)$ such that for any automorphism φ of G, either $\varphi(X) = X$ or $\varphi(X) \cap X = \emptyset$.

Theorem 2.4 (Tindell,1982)

A connected vertex-transitive graph G with degree $k \geq 3$ is super-λ if and only if there is no imprimitive block of G which is a clique of size k.

R. Tindell, Edge connectivity properties of symmetric graphs, Preprint, Stevens Institute of Technology, Hoboken, NJ, 1982.

Restricted edge-connectivity

For a connected graph G, an edge-cut F of G is said to be an s-restricted edge-cut if every component of $G - F$ has at least s vertices. The s-restricted edge-connectivity of G is the minimum cardinality over all s-restricted edge-cuts of G, denoted by $\lambda_s(G)$.

Let $\xi(G)$ be the minimum edge-degree of G. Esfahanian and Hakimi (1988) showed that if a connected graph G of oder n is not a star $K_{1,n-1}$, then $\lambda_2(G)$ is well-defined and $\lambda_2(G) \leq \xi(G)$. A connected graph G is called to be maximally restricted edge-connected, if $\lambda_2(G) = \xi(G)$.

Furthermore, a maximally restricted edge-connected graph G is called to be super restricted edge-connected, in short, super-λ_2, if every minimum 2-restricted edge-cut of G isolates an edge.

Theorem 2.5 (Xu, 2000)

Let G be a connected vertex-transitive graph of order at least 4. Then G is maximally restricted edge-connected if its order is odd or it has no triangle.

Wang studied the super restricted edge-connectivity of connected vertex-transitive graphs. The girth of a graph G is the length of a shortest cycle in G.

Theorem 2.6 (Wang, 2004)

If G is a connected vertex-transitive graph with degree $k > 2$ and girth $g > 4$, then it is super-λ_2.

Junming Xu, Restricted edge-connectivity of vertex-transitive graphs, Chinese Ann. Math. Ser. A 21 (2000) 605-608.

Yingqian Wang, Super restricted edge-connectivity of vertex-transitive graphs, Discrete Math. 289 (2004) 199-205.

Ou and Zhang studied the 3-restricted edge-connectivity of vertex-transitive graphs and proved the following results.

Lemma 2.7 (Ou and Zhang, 2005)

If G is a connected k-regular vertex-transitive graph of order at least 6 and girth $g \geq 4$, then either $\lambda_3 = 3k - 4$ or λ_3 is a divisor of $|V(G)|$ such that $2k - 2 \leq \lambda_3 \leq 3k - 5$ unless $k = 3$ and $g = 4$.

Jianping Ou, Fuji Zhang, 3-restricted edge connectivity of vertex transitive graphs. Ars Combin. 74 (2005), 291 – 301.

An improvement on Wang's result

Using Lemma 2.7, we make an improvement on Wang's result (Lemma 2.6) for vertex-transitive odd graphs.

Lemma 2.8

If G is a connected vertex-transitive odd graph with degree $k > 2$ and girth $g > 3$, then it is super-λ_2.

Proof.

By Lemma 2.7, either $\lambda_3(G) = 3k - 4$ or $\lambda_3(G)$ is a divisor of $|V(G)|$. Noting that G is regular and $|V(G)|$ is odd, k is even and $k \geq 4$. Then $d(Y) = k|Y| - 2|E(G[Y])|$ is even for any $Y \subseteq V(G)$, implying $\lambda_3(G)$ is even. Thus $\lambda_3(G) = 3k - 4$. Note that a connected graph H with a restricted edge-cut is super-λ_2 if and only if either H has no 3-restricted edge-cut or $\lambda_3(H) > \xi(H)$. Thus, G is super-λ_2 since $\lambda_3(G) = 3k - 4 > 2k - 2 = \xi(G)$. □

This short proof is pointed out by Prof. Junming Xu.

Cyclic edge-connectivity

For a connected graph G, an edge-cut F of G is called a cyclic edge-cut if at least two components of $G - F$ contain cycles. The cyclic edge-connectivity of G with a cyclic edge-cut is defined as the minimum cardinality over all cyclic edge-cuts of G, denoted by $\lambda_c(G)$.

Theorem 2.9 (Wang and Zhang, 2009)

Let G be a connected vertex-transitive graph with degree $k \geq 4$ and girth $g \geq 5$. Then $\lambda_c(G) = (k-2)g$.

Bing Wang, Zhao Zhang, On cyclic edge-connectivity of transitive graphs, Discrete Math. 309 (2009) 4555-4563.

A useful lemma

A subset X of $V(E)$ is called an independent set of a graph G if $E(G[X]) = \emptyset$. The independent number of G is the maximum cardinality of an independent set of G, denoted by $\alpha(G)$.

Lemma 3.1

Let G be a connected vertex-transitive odd graph with degree $k \geq 4$. Then $\alpha(G) < (|V(G)| - 1)/2$.

Theorem 1.6: A connected vertex-transitive odd graph G of order at least 5 is 3-factor-critical if and only if G has the regularity degree $k \geq 4$.

The Proof Sketch for Theorem 1.6

The "only if" part is trivial by Lemma 1.5. Next we will finish the "if" part.

Suppose that G is not a cycle ($k \geq 4$) and is not 3-factor-critical. Note that G is factor-critical by Theorem 1.2(a). By Lemma 1.3, there is a set $X \subseteq V(G)$ with $|X| \geq 3$ such that ($c_0(G-X)$ denotes the number of odd components of $G-X$)

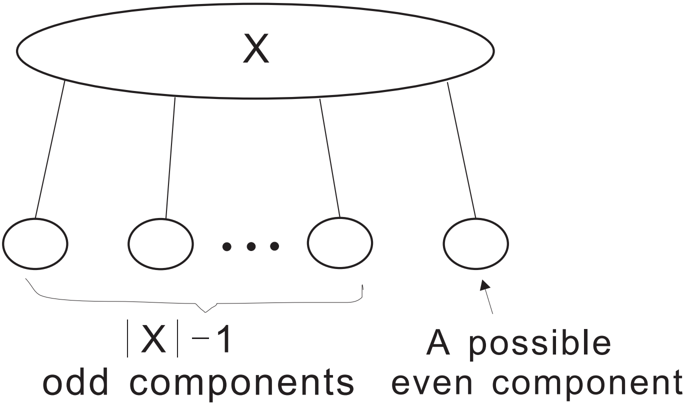
$$|X| - 3 < c_0(G-X) \leq |X| - 1.$$

Since $c_0(G-X)$ and $|X|$ have different parity,

$$c_0(G-X) = |X| - 1.$$

Let H_1, H_2, ..., H_t be the odd components of $G-X$ where $t = c_0(G-X)$.

The Proof Sketch for Theorem 1.6 (continued)



The Proof Sketch for Theorem 1.6 (continued)

Noting that G is odd and is not a cycle, the degree k of G is even and $k \geq 4$. It follows that there is no imprimitive block of G which is a clique of size k. By Lemma 2.4, G is super-λ.

Claim 1. Every component of $G - X$ is odd.

If $G - X$ has an even component H_0, then $d(V(H_i)) \geq k$ for $0 \leq i \leq t$ since $\lambda(G) = k$ by Lemma 2.3. Thus,

$$k|X| = k(t+1) \leq \sum_{i=0}^{t} d(V(H_i)) \leq d(X) \leq k|X|,$$

which implies that $d(V(H_0)) = k$ and X is an independent set of G. Hence $\nabla(V(H_0))$ isolates a vertex v in G since G is super-λ, and $v \in X$. This means that $G[V(H_0) \cup \{v\}]$ is a component of G, a contradiction. Claim 1 holds.

The Proof Sketch for Theorem 1.6 (continued)

Claim 2. If $g \geq 4$, then $G - X$ has exactly one nontrivial component H, and $d(V(H)) = 2k$.

Suppose that $g \geq 4$. Assume that H_1, H_2, …, H_p are nontrivial components and H_{p+1}, H_{p+2}, …, H_t are singletons. Note that G is super-λ_2 by Theorem 2.7 and is maximally edge-connected by Lemma 2.3. It is not difficult to show that $d(V(H_i)) > 2k - 2$ for $i = 1, 2, \ldots, p$. We have

$$p(2k-2) + k(t-p) < \sum_{i=1}^{p} d(V(H_i)) + k(t-p) = \sum_{i=1}^{t} d(V(H_i)) = d(X) \leq k|X|.$$

Note that $t = c_0(G - X) = |X| - 1$. It follows that $p < \frac{k}{k-2} \leq 2$.

If $p = 0$, then $\overline{X}$ is an independent set of size $(|V(G)| - 1)/2$ in G, which contradicts that $\alpha(G) < (|V(G)| - 1)/2$ by Lemma 3.1.

So $p = 1$. Then $2k - 2 < d(V(H_1)) \leq 2k$. Since $d(V(H_1)) = k|V(H_1)| - 2|E(H_1)|$ is even, $d(V(H_1)) = 2k$. Claim 2 is proved.

The Proof Sketch for Theorem 1.6 (continued)

Claim 3. $g \geq 4$.

If $g = 3$, using the connectivity and super edge-connectivity of vertex-transitive odd graphs, we can show that the number of singletons in $G - X$ is more than the number of edges in $G[X]$, which will contradict the following lemma.

Lemma 3.2

Let G be a vertex-transitive graph with a triangle. Then, for each subset $X \subseteq V(G)$, the number of singletons in $G - X$ is not more than the number of edges in $G[X]$.

The Proof Sketch for Theorem 1.6 (continued)

Claim 4. $g = 4$ **and** $k = 4$**.**

Suppose that $g \geq 4$. Since k is even, $d(Y) = k|Y| - 2|E(G[Y])|$ is even for any $Y \subseteq V(G)$. Hence $\lambda_3(G)$ is even. By Lemma 2.8, $\lambda_3(G) = 3k - 4$.

Firstly, it is not difficult to show that $\nabla(V(H))$ is a 3-restricted edge cut of G. Then

$$2k = d(V(H)) \geq \lambda_3(G) = 3k - 4,$$

which implies that $k = 4$.

Secondly, it is also not difficult to show that $\nabla(V(H))$ is a cyclic edge-cut of G. Then

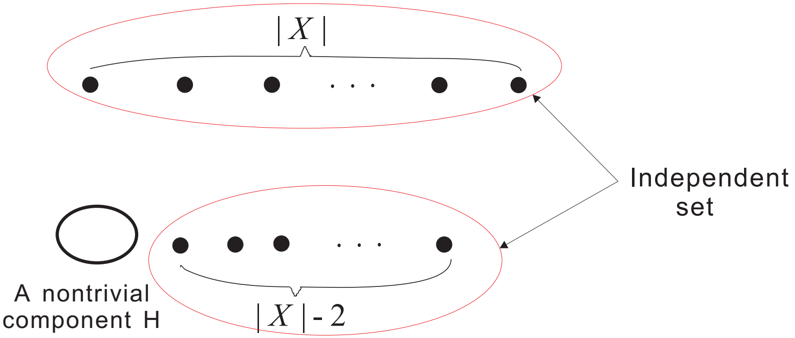
$$2k = d(V(H)) \geq \lambda_c(G) = (k-2)g,$$

It follows that $g \leq \frac{2k}{k-2} = 4$.

Claim 4 holds.

The Proof Sketch for Theorem 1.6 (continued)

From Claim 4, $g = 4$ and $k = 4$.



Lemma 3.3

Let G be a connected 4-regular vertex-transitive triangle-free odd graph. Then G has no two distinct vertices u and v such that $N(u) = N(v)$.

Lemma 3.4

Let G be a connected vertex-transitive odd graph with degree $k = 4$ and girth $g = 4$. Then, for each edge e in G, there are at least two distinct quadrangles containing e and there is another edge e' adjacent to e such that the number of quadrangles containing e' is equal to the number of quadrangles containing e.

We can show that either G has two distinct vertices u and v such that $N(u) = N(v)$ which contradicts Lemma 3.3, or for every vertex w there are two edges incident to w, one contained in two quadrangles and another contained in at least six quadrangles, by Lemma 3.4. On the other hand, we can find that for any 4-regular graph of girth 4 the second case is false. So a contradiction exists inevitably.

By discussing high order restricted edge-connectivity of vertex-transitive graphs, we obtain a further result about the p-factor-criticality of vertex-transitive graphs.

$\xi_s(G) = \min\{d(X) : X \subseteq V(G) \text{ such that } |X| = s \text{ and } G[X] \text{ is connected}\}$, where $d(X)$ is the number of edges with exactly one end in X. A connected graph G is called super s-restricted edge-connected, or simply super-λ_s, if $\lambda_s(G) = \xi_s(G)$ and every minimum s-restricted edge-cut of G isolates one component $G[X]$ with $|X| = s$.

We obtain the following result about high order restricted edge-connectivity of vertex-transitive graphs.

Theorem 4.1

Let G be a connected vertex-transitive graph with degree $k > 5$ and girth $g > 5$. Then G is super-λ_s for any positive integer s with $s \leq 2g$ when $kg > 36$ and for any positive integer s with $s \leq 2g - 2$ when $kg = 36$.

Using Theorem 4.1, we show that a connected vertex-transitive non-bipartite graph with large girth is p-factor-critical.

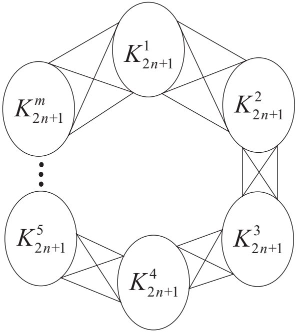
Theorem 4.2

Let G be a connected vertex-transitive non-bipartite graph with degree $k \geq 6$ and girth $g \geq 6$ and p be a positive integer with the same parity as $|V(G)|$, $p \leq k-1$. If g also satisfies $g \geq \frac{k(p+1)}{2(k-2)}$, then G is p-factor-critical.

If we relax the restriction on the girth, then there is such vertex-transitive graph G that G satisfies the necessary condition ($\kappa(G) \geq p$ and $\lambda(G) \geq p+1$) of p-factor-critical graphs in Lemma 1.5 but is not p-factor-critical.

Example. Let H_1 and H_2 be two graphs. The lexicographic product G of H_1 and H_2, denoted by $G = H_1 \circ H_2$, is defined as follows:
$V(G) = V(H_1) \times V(H_2)$, and $[(x_1, x_2), (y_1, y_2)] \in E(G)$ if and only if $(x_1, y_1) \in E(H_1)$ or $x_1 = y_1$ and $(x_2, y_2) \in E(H_2)$.

Let $G = C_m \circ K_{2n+1}$, where $m \geq 4$. Then G is vertex-transitive, the vertex-connectivity $\kappa(G) = 4n + 2$ and edge-connectivity $\lambda(G) = 6n + 2$. But G is not $(4n + 2)$-factor-critical.



THANK YOU!